The Cutoff Structure of Top Trading Cycles in School Choice

Supplementary Appendix

D Proofs for Section 3

D.1 Definitions and Notation

We begin with some additional definitions and notation.

Let $\underline{x}, \overline{x}$ be vectors. We let $(\underline{x}, \overline{x}] = \{x : x \not\leq \underline{x} \text{ and } x \leq \overline{x}\}$ denote the set of vectors that are weakly smaller than $\overline{x}$ along every coordinate, and strictly larger than $\underline{x}$ along some coordinate. Let $K \subseteq \mathcal{C}$ be a set of schools. For all vectors x, we let $\pi_K(x)$ denote the projection of x to the coordinates indexed by schools in K.

We now incorporate information about the set of available schools. We denote by

$$\Theta^{c|C} = \left\{\theta \in \Theta | \, Ch^{\theta}(C) = c\right\}$$

the set of students whose top choice in C is c, and denote by $\eta^{c|C}$ the measure of these students. That is, for $S \subseteq \Theta$, let $\eta^{c|C}(S) := \eta\left(S \cap \Theta^{c|C}\right)$. In an abuse of notation, for a set $A \subseteq [0,1]^{\mathcal{C}}$, we will often let $\eta(A)$ denote $\eta\left(\left\{\theta \in \Theta \,|\, r^{\theta} \in A\right\}\right)$, the measure of students with ranks in A, and let $\eta^{c|C}(A)$ denote $\eta\left(\left\{\theta \in \Theta^{c|C} \,|\, r^{\theta} \in A\right\}\right)$, the measure of students with ranks in A whose top choice school in C is c.

We will also find it convenient to define sets of students who were offered or assigned a seat along some TTC path γ. These will be useful in considering the result of aggregating the marginal trade balance equations. For each time τ let $\mathcal{T}_c(\gamma;\tau) \overset{def}{=} \{\theta \in \Theta \mid \exists \tau' \leq \tau \text{ s.t. } r^{\theta}_c = \gamma_c(\tau') \text{ and } r^{\theta} \leq \gamma(\tau')\}$ denote the set of students who were offered a seat by school c before time τ, let $\mathcal{T}^c(\gamma;\tau) \overset{def}{=} \{\theta \in \Theta \mid r^{\theta} \not\leq \gamma(\tau) \text{ and } Ch^{\theta}\left(C\left(r^{\theta}\right)\right) = c\}$ denote the set of students who were assigned a seat at school c before time τ, and let $\mathcal{T}^{c|C}(\gamma;\tau) \overset{def}{=} \{\theta \in \Theta \mid r^{\theta} \not\leq \gamma(\tau) \text{ and } Ch^{\theta}(C) = c\}$ denote the set of students who would be assigned a seat at school c before time τ if the set of available schools was C and the path followed was γ.[31]

[31]Note that $\mathcal{T}_c(\gamma;\tau)$ and $\mathcal{T}^c(\gamma;\tau)$ include students who were offered or assigned a seat in the school in previous rounds.

For each interval $T = [\underline{t}, \overline{t}]$ let $\mathcal{T}_c(\gamma; T) \overset{def}{=} \mathcal{T}_c\left(\gamma; \overline{t}\right) \setminus \mathcal{T}_c(\gamma; \underline{t})$ be the set of students who were offered a seat by school c at some time $\tau \in T$, and let $\mathcal{T}^{c|C}(T;\gamma) \overset{def}{=} \mathcal{T}^{c|C}\left(\gamma; \overline{t}\right) \setminus \mathcal{T}^{cC}(\gamma; \underline{t})$ be the set of students who were assigned to a school c at some time $\tau \in T$, given that the set of available schools was $C(\gamma(\tau)) = C$ for each $\tau \in T$. For each union of disjoint intervals $T = \cup_n T_n$ similar define $\mathcal{T}_c(\gamma; T) \overset{def}{=} \cup_n \mathcal{T}_c(\gamma; T_n)$ and $\mathcal{T}^{c|C}(T;\gamma) \overset{def}{=} \cup_n \mathcal{T}^{c|C}(T_n;\gamma)$.

Finally let us set up the definitions for solving the marginal trade balance equations. For a set of schools C and individual schools $b, c \in C$, recall that

$$\begin{aligned} H_b^{c|C}(x) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta\left(\left\{\theta \in \Theta \mid r^\theta \in \left[x - \varepsilon \cdot e^b, x\right) \text{ and } Ch^\theta(C) = c\right\}\right) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta\left(\left\{\theta \in \Theta^{c|C} \mid r^\theta \in \left[x - \varepsilon \cdot e^b, x\right)\right\}\right) \end{aligned}$$

is the marginal density of students pointed to by school b at the point x whose top choice school in C is c.

Let $\widetilde{H}^C(x)$ be the $|C| \times |C|$ matrix with (b,c)th entry $\widetilde{H}^C(x)_{b,c} = \frac{1}{\overline{v}} H_b^{c|C}(x) + \mathbf{1}_{b=c}\left(1 - \frac{v_c}{\overline{v}}\right)$, where $v_c = \sum_{d \in C} H_c^{d|C}(x)$ is the row sum of $H(x)$, and the normalization $\overline{v}$ satisfies $\overline{v} \geq \max_c v_c$.

Let $M^C(x)$ be the Markov chain with state space C, and transition probability from state b to state c equal to $\widetilde{H}^C(x)_{b,c}$. We remark that such a Markov chain exists, since $\widetilde{H}^C(x)$ is a (right) stochastic matrix for each pair C, x.

We will also need the following definitions. For a matrix H and sets of indices I, J we let $H_{I,J}$ denote the submatrix of H with rows indexed by elements of I and columns indexed by elements of J. Recall that, by Assumption 1, the measure η is defined by a probability density ν that is right-continuous and piecewise Lipschitz continuous with points of discontinuity on a finite grid. Let the finite grid be the set of points $\{x \mid x_i \in D_i \forall i\}$, where the D_i are finite subsets of $[0,1]$. Then there exists a partition $\mathcal{R}$ of $[0,1]^{\mathcal{C}}$ into hyperrectangles such that for each $R \in \mathcal{R}$ and each face of R, there exists an index i and $y_i \in D_i$ such that the face is contained in $\{x \mid x_i = y_i\}$.

The following notion of continuity will be useful, given this grid-partition. We say that a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ is *right-continuous* if $f(x) = \lim_{y \geq x} f(y)$, where x, y are vectors in $\mathbb{R}^n$ and the inequalities hold coordinate-wise. For an $m \times n$

matrix A, let $\mathbf{1}(A)$ be the $m \times n$ matrix with entries

$$\mathbf{1}(A)_{ij} = \begin{cases} 1 & \text{if } A_{ij} \neq 0, \\ 0 & \text{if } A_{ij} = 0. \end{cases}$$

We will also frequently make use of the following lemmas.

Lemma 2. *Let γ satisfy the marginal trade balance equations. Then γ is Lipschitz continuous.*

Proof. By assumption, γ is normalized so that $\|\frac{d\gamma(t)}{dt}\|_1 = 1$ a.e., and so since $\gamma(\cdot)$ is monotonically decreasing, for all c it holds that $\gamma_c(\cdot)$ has bounded derivative and is Lipschitz with Lipschitz constant L_c. It follows that $\gamma(\cdot)$ is Lipschitz with Lipschitz constant $\max_c L_c$. $\square$

Lemma 3. *Let $C \subseteq \mathcal{C}$ be a set of schools, and let D be a region on which $\widetilde{H}^C(x)$ is irreducible for all $x \in D$. For each x let $A(x)$ be given by replacing the nth column of $\widetilde{H}^C(x) - I_C$ with the all ones vector $\mathbf{1}$.*[32] *Then the function $f(x) = \begin{bmatrix} \mathbf{0}^T & 1 \end{bmatrix} A(x)^{-1}$ is piecewise Lipschitz continuous in x.*

Proof. It suffices to show that the function which, for each x, outputs the matrix $A(x)^{-1}$ is piecewise Lipschitz continuous in x.

Now

$$H_b^{c|C}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\theta : r^\theta \geq x, r^\theta \not\geq x_b + \varepsilon \cdot e_b, c \succ^\theta C} \nu(\theta)\, d\theta,$$

where $\nu(\cdot)$ is bounded below on its support and piecewise Lipschitz continuous, and the points of discontinuity lie on the grid. Hence $H_b^{c|C}(x)$ is Lipschitz continuous in x for all b, c, and $\sum_d H_c^{d|C}(x)$ nonzero and hence bounded below, and so $\widetilde{H}^C(x)_{b,c}$ is bounded above and piecewise Lipschitz continuous in x, and therefore so is $A(x)$. Finally, since $\widetilde{H}^C(x)$ is an irreducible row stochastic matrix for each $x \in D$, it follows that $A(x)$ is full rank and continuous. This is because when $\widetilde{H}^C(x)$ is irreducible every choice of $n-1$ columns of $\widetilde{H}^C(x) - I_C$ gives an independent set whose span does not contain the all ones vector $\mathbf{1}_C$. Therefore if we let $A(x)$ be given by replacing the nth column in $\widetilde{H}^C(x) - I_C$ with $\mathbf{1}_C$, then $A(x)$ has full rank.

Since $A(x)$ is full rank and continuous, in each piece $det(A(x))$ is bounded away from 0, and so $A(x)^{-1}$ is piecewise Lipschitz continuous, as required. $\square$

[32] I_C is the identity matrix with rows and columns indexed by the elements in C.

D.2 Connection to Continuous Time Markov Chains

In Section A.3, we appealed to a connection with Markov chain theory to provide a method for solving for all the possible values of $\boldsymbol{d}(x)$. Specifically, we argued that if $\mathcal{K}(x)$ is the set of recurrent communication classes of $\widetilde{H}(x)$, then the set of valid directions $\boldsymbol{d}(x)$ is identical to the set of convex combinations of $\left\{\boldsymbol{d}^K\right\}_{K\in\mathcal{K}(x)}$, where $\boldsymbol{d}^K$ is the unique solution to the trade balance equations (1) restricted to K. We present the relevant definitions, results and proofs here in full.

Let us first present some definitions from Markov chain theory.[33] A square matrix P is a *right-stochastic matrix* if all the entries are non-negative and each row sums to 1. A *probability vector* is a vector with non-negative entries that add up to 1. Given a right-stochastic matrix P, the *Markov chain with transition matrix* P is the Markov chain with state space equal to the column/row indices of P, and a probability P_{ij} of moving to state j in one time step, given that we start in state i. Given two states i, j of a Markov chain with transition matrix P, we say that states i and j *communicate* if there is a positive probability of moving to state i to state j in finite time, and vice versa.

For each Markov chain, there exists a unique decomposition of the state space into a sequence of disjoint subsets $C_1, C_2, \ldots$ such that for all i, j, states i and j communicate if and only if they are in the same subset C_k for some k. Each subset C_k is called a *communication class* of the Markov chain. A Markov chain is *irreducible* if it only has one communication class. A state i is *recurrent* if, starting at i and following the transition matrix P, the probability of returning to state i is 1. A communication class is recurrent if it contains a recurrent state.

The following proposition gives a characterization of the stationary distributions of a Markov chain. We refer the reader to any standard stochastic processes textbook (e.g. Karlin and Taylor (1975)) for a proof of this result.

Proposition 10. *Suppose that P is the transition matrix of a Markov chain. Let $\mathcal{K}$ be the set of recurrent communication classes of the Markov chain with transition matrix P. Then for each recurrent communication class $K \in \mathcal{K}$, the equation $\pi = \pi P$ has a unique solution π^K such that $||\pi^K|| = 1$ and supp $\left(\pi^K\right) \subseteq K$. Moreover, the support of π^K is equal to K. In addition, if $||\pi|| = 1$ and π is a solution to the equation $\pi = \pi P$, then π is a convex combination of the vectors in $\{\pi^K\}_{K\in\mathcal{K}}$.*

[33] See standard texts such as Karlin and Taylor (1975) for a more complete treatment.

To make use of this proposition, define at each point x and for each set of schools C a Markov chain $M^C(x)$ with transition matrix $H^C(x)$. We will relate the valid directions $\boldsymbol{d}(x)$ to the recurrent communication classes of $M^C(x)$, where C is the set of available schools. We will need the following notation and definitions. Given a vector v indexed by $\mathcal{C}$, a matrix Q with rows and columns indexed by C and subsets $K, K' \subseteq C$ of the indices, we let v_K denote the restriction of v to the coordinates in K, and we let $Q_{K,K'}$ denote the restriction of Q to rows indexed by K and columns indexed by K'.

The following lemma characterizes the recurrent communication classes of the Markov chain $M^C(x)$ using the properties of the matrix $H^C(x)$, and can be found in any standard stochastic processes text.

Lemma 4. *Let C be the set of available school at a point x. Then a set $K \subseteq C$ is a recurrent communication class of the Markov chain $M^C(x)$ if and only if $H^C(x)_{K,K}$ is irreducible and $H^C(x)_{K,C\setminus K}$ is the zero matrix.*

Proposition 10 and Lemma 4 allow us to characterize the valid directions $d(x)$.

Theorem 4. *Let C be the set of available schools, and let $\mathcal{K}(x)$ be the set of subsets $K \subseteq C$ for which $\widetilde{H}^C(x)_{K,K}$ is irreducible and $\widetilde{H}^C(x)_{K,C\setminus K}$ is the zero matrix. Then the equation $\boldsymbol{d} = \boldsymbol{d} \cdot \widetilde{H}^C(x)$ has a unique solution $\boldsymbol{d}^K$ that satisfies $\boldsymbol{d}^K \cdot \mathbf{1} = -1$ and $\operatorname{supp}\left(\boldsymbol{d}^K\right) \subseteq K$, and its projection onto its support K has the form*

$$\left(\boldsymbol{d}^K\right)_K = \left[\begin{array}{cc} \mathbf{0}^T & -1 \end{array}\right] A_K^C(x)^{-1},$$

where $A_K^C(x)$ is the matrix obtained by replacing the $(|K|-1)$th column of $\widetilde{H}^C(x)_{K,K} - I_K$ with the all ones vector $\mathbf{1}_K$.

Moreover, if $\boldsymbol{d} \cdot \mathbf{1} = -1$ and d is a solution to the equation $\boldsymbol{d} = \boldsymbol{d} \cdot \widetilde{H}^C(x)$, then $\boldsymbol{d}$ is a convex combination of the vectors in $\{\boldsymbol{d}^K\}_{K\in\mathcal{K}(x)}$.

Proof. Proposition 4 shows that the sets K are precisely the recurrent sets of the Markov chain with transition matrix $\widetilde{H}(x)$. Hence uniqueness of the d^K and the fact that d is a convex combination of d^K follow directly from Proposition 10. The form of the solution d^K follows from Theorem 5. □

This has the following interpretation. Suppose that there is a unique recurrent communication class K, such as when η has full support. Then there is a unique infinitesimal continuum trading cycle of students, specified by the unique direction $\boldsymbol{d}$ satisfying $\boldsymbol{d} = \boldsymbol{d} \cdot \widetilde{H}(x)$. Moreover, students in the cycle trade seats from every school in K. Any school not in K is blocked from participating, since there is not enough demand to fill the seats they are offering. When there are multiple recurrent communication classes, each of the $\boldsymbol{d}^K$ gives a unique infinitesimal trading cycle of students, corresponding to those who trade seats in K. Moreover, these trading cycles are disjoint. Hence the only multiplicity that remains is to decide the order, or the relative rate, at which to clear these cycles. We will show in Section D.3 that, as in the discrete setting, the order in which cycles are cleared does not affect the final allocation.

D.3 Proof of Theorem 2

We first show that there exist solutions p, γ, t to the marginal trade balance equations and capacity equations. The proof relies on selecting appropriate valid directions $\boldsymbol{d}(x)$ and then invoking the Picard-Lindelöf theorem to show existence.

Specifically, let C be the set of available schools, fix a point x, and consider the set of vectors $\boldsymbol{d}$ such that $\boldsymbol{d} \cdot H^C(x) = \boldsymbol{d}$. Then it follows from Theorem 4 that if $\boldsymbol{d}(x)$ is the valid direction from x with minimal support under the shortlex order, then $\boldsymbol{d}(x) = \boldsymbol{d}^{K(x)}$ for the element $K(x) \in \mathcal{K}(x)$ that is the smallest under the shortlex ordering. As the density $\nu(\cdot)$ defining $\eta(\cdot)$ is Lipschitz continuous, it follows that $\mathcal{K}(\cdot)$ and $K(\cdot)$ are piecewise constant. Hence we may invoke Lemma 3 to conclude that $\boldsymbol{d}(\cdot)$ is piecewise Lipschitz within each piece, and hence piecewize Lipschitz in $[0,1]^{\mathcal{C}}$.Since $\boldsymbol{d}(\cdot)$ is piecewise Lipschitz, it follows from the Picard-Lindelöf theorem that there exists a unique function $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = \boldsymbol{d}(\gamma(t))$. It follows trivially that γ satisfies the marginal trade balance equations, and since we have assumed that all students find all schools acceptable and there are more students than seats it follows that there exist runout times $t^{(c)}$ and cutoffs p_b^c.

Proof of Uniqueness

In this section, we prove part (ii) of Theorem 2, that any two valid TTC paths give equivalent allocations. The intuition for the result is the following. The connection to Markov chains shows that having multiple possible valid direction in the continuum is parallel to having multiple possible trade cycles in the discrete model. Hence the only multiplicity in choosing valid TTC directions is whether to implement one set of trades before the others, or to implement them in parallel at various relative rates. We can show that the set of cycles is independent of the order in which cycles are selected, or equivalently that the sets of students who trade with each other is independent of the order in which possible trades are executed. It follows that any pair of valid TTC paths give the same final allocation.

We remark that the crux of the argument is similar to what shows that discrete TTC gives a unique allocation. However, the lack of discrete cycles and the ability to implement sets of trades in parallel both complicate the argument and lead to a rather technical proof.

We first formally define *cycles* in the continuum setting, and a partial order over the cycles corresponding to the order in which cycles can be cleared under TTC. We then define the set of cycles $\Sigma(\gamma)$ associated with a valid TTC path γ. Finally, we show that the sets of cycles associated with two valid TTC paths γ and γ' are the same, $\Sigma(\gamma) = \Sigma(\gamma')$.

Definition 3. A *(continuum) cycle* $\sigma = (K, \underline{x}, \overline{x})$ is a set $K \subseteq \mathcal{C}$ and a pair of vectors $\underline{x} \leq \overline{x}$ in $[0,1]^{\mathcal{C}}$. The cycle σ is *valid* for available schools $\{C(x)\}_{x \in [0,1]^{\mathcal{C}}}$ if $K \in \mathcal{K}^{C(x)}(x)\, \forall x \in (\underline{x}, \overline{x}]$.

Intuitively, a cycle is defined by two time points in a run of TTC, which gives a set of students,[34] and the set of schools they most desire. A cycle is valid if the set of schools involved is a recurrent communication class of the associated Markov chains. We say that a cycle $\sigma = (K, \underline{x}, \overline{x})$ *appears* at time t in $TTC(\gamma)$ if $K \in \mathcal{K}^{C(\gamma(t))}(\gamma(t))$ and $\gamma_c(t) = \overline{x}_c$ for all $c \in K$. We say that a *student* θ *is in cycle* σ if $r^\theta \in (\underline{x}, \overline{x}]$[35], and a *school* c *is in cycle* σ if $c \in K$.

Definition 4 (Partial order over cycles)**.** The cycle $\sigma = (K, \underline{x}, \overline{x})$ *blocks* the cycle $\sigma' = (K', \underline{x}', \overline{x}')$, denoted by $\sigma \rhd \sigma'$, if at least one of the following hold:

(Blocking student) There exists a student θ in σ' who prefers a school in K to all those in K', that is, there exists θ and $c \in K \setminus K'$ such that $c \succ^\theta c'$ for all $c' \in K'$.

(Blocking school) There exists a school in σ' that prefers a positive measure of students in σ to all those in σ', that is, there exists $c \in K'$ such that $\eta\left(\theta \,|\, \theta \text{ in } \sigma,\, r_c^\theta > \overline{x}'_c\right) >$

[34]The set of students is given by taking the difference between two nested hyperrectangles, one with upper coordinate $\overline{x}$ and the other with upper coordinate $\underline{x}$.

[35]Recall that since $r^\theta, \underline{x}$ and $\overline{x}$ are vectors, this is equivalent to saying that $r^\theta \not\leq \underline{x}$ and $r^\theta \leq \overline{x}$.

0.[36]

Let us now define the set of cycles associated with a run of TTC. We begin with some observations about $H_c^{b|C}(\cdot)$ and $\widetilde{H}^C(\cdot)_{bc}$. For all $b, c \in C$ the function $H_c^{b|C}(\cdot)$ is right-continuous on $[0,1]^{\mathcal{C}}$, Lipschitz continuous on R for all $R \in \mathcal{R}$ and uniformly bounded away from zero on its support. Hence $\mathbf{1}\left(H_c^{b|C}(\cdot)\right)$ is constant on R for all $R \in \mathcal{R}$. It follows that $\widetilde{H}^C(\cdot)_{bc}$ is also right-continuous, and Lipschitz continuous on R for all $R \in \mathcal{R}$. Moreover, there exists some finite rectangular subpartition $\mathcal{R}'$ of $\mathcal{R}$ such that for all $C \subseteq \mathcal{C}$ the function $\mathbf{1}\left(\widetilde{H}^C(\cdot)\right)$ is constant on R for all $R \in \mathcal{R}'$.

Definition 5. The partition $\mathcal{R}'$ is the minimal rectangular subpartition of $\mathcal{R}$ such that for all $C \subseteq \mathcal{C}$ the function $\mathbf{1}\left(\widetilde{H}^C(\cdot)\right)$ is constant on R for all $R \in \mathcal{R}'$.

For $x \in [0,1]^{\mathcal{C}}$ and $C \subseteq \mathcal{C}$, let $\mathcal{K}^C(x)$ be the recurrent communication classes of the Markov chain $M^C(x)$. The following lemma follows immediately from Proposition 4, since $\mathbf{1}\left(\widetilde{H}^C(\cdot)\right)$ is constant on $R\, \forall R \in \mathcal{R}'$, and recurrent communication classes depend only on $\mathbf{1}\left(\widetilde{H}^C\right)$.

Lemma 5. *$\mathcal{K}^C(\cdot)$ is constant on R for every $R \in \mathcal{R}'$.*

For each $K \in \mathcal{K}^C(x)$, let $d^K(x)$ be the unique vector satisfying $d = d\widetilde{H}^C(x)$, which exists by Theorem 4.

Let γ be a TTC path, and assume that the schools are indexed such that for all x there exists ℓ such that $C(x) = \mathcal{C}^{(\ell)} \stackrel{def}{=} \{\ell, \ell+1, \ldots, |\mathcal{C}|\}$. For each set of schools $K \subseteq \mathcal{C}$, let $T^{(\ell)}(K,\gamma)$ be the set of times τ such that $C(\gamma(\tau)) = \mathcal{C}^{(\ell)}$ and K is a recurrent communication class for $\tilde{H}^{\mathcal{C}^{(l)}}(\gamma(\tau))$. Since γ is continuous and weakly decreasing,it follows from Lemma (5) that $T^{(\ell)}(K,\gamma)$ is the finite disjoint union of intervals of the form $\left[\underline{t}, \overline{t}\right)$. Let $\mathcal{I}\left(T^{(\ell)}(K,\gamma)\right)$ denote the set of intervals in this disjoint union. We may assume that for each interval T, $\gamma(T)$ is contained in some hyperrectangle $R \in \mathcal{R}'$.[37]

For a time interval $T = \left[\underline{t}, \overline{t}\right) \in \mathcal{I}\left(T^{(\ell)}(K,\gamma)\right)$, we define the cycle $\sigma(T) = (K, \underline{x}(T), \overline{x}(T))$ as follows. Intuitively, we want to define it simply as $\sigma(T) = \left(K, \gamma(\underline{t}), \gamma(\overline{t})\right)$, but in order to minimize the dependence on γ, we define the endpoints $\underline{x}(T)$ and $\overline{x}(T)$ of the interval of ranks to be as close together as possible,

[36] We note that it is necessary but not sufficient that $\overline{x}_c > \overline{x}'_c$.

[37] This is without loss of generality, since if $\gamma(T)$ is not contained we can simply partition T into a finite number of intervals $\cup_{R\in\mathcal{R}'}\gamma^{-1}(\gamma(T) \cap R)$, each contained in a hyperrectangle in $\mathcal{R}'$.

while still describing the same set of students (up to a set of η-measure 0). Define

$$\begin{aligned}\underline{x}(T) &= \max\left\{x : \gamma(\underline{t}) \leq x \leq \gamma(\underline{t}) \ , \ \eta\left(\theta : Ch_\theta\left(\mathcal{C}^{(\ell)}\right) \in K, \ r^\theta \in \left(x, \gamma\left(\bar{t}\right)\right]\right) = 0\right\}, \\ \overline{x}(T) &= \min\left\{x : \gamma(\underline{t}) \leq x \leq \gamma(\underline{t}) \ : \ \eta\left(\theta : Ch_\theta\left(\mathcal{C}^{(\ell)}\right) \in K, \ r^\theta \in \left(\gamma(\underline{t}), x\right]\right) = 0\right\},\end{aligned}$$

to be the points chosen to be maximal and minimal respectively such that the set of students allocated by γ during the time interval T has the same η-measure as if $\gamma(\underline{t}) = \underline{x}(\tau)$ and $\gamma\left(\bar{t}\right) = \overline{x}(\tau)$. In other words $\eta\left(\left(\cup_{c\in K}\mathcal{T}^c\left(\gamma;\bar{t}\right) \setminus \mathcal{T}^c\left(\gamma;\underline{t}\right)\right) \setminus \left\{\theta : Ch_\theta\left(\mathcal{C}^{(\ell)}\right) \in K, \ r^\theta \in \left(\underline{x}(T), \overline{x}(T)\right]\right\}\right) = 0.$

In a slight abuse of notation, if $\sigma = \sigma(T)$ we will let $\underline{x}(\sigma)$ denote $\underline{x}(T)$ and $\overline{x}(\sigma)$ denote $\overline{x}(T)$.

Definition 6. The *set of cycles cleared by $TTC(\gamma)$ in round ℓ*, denoted by $\Sigma^{(\ell)}(\gamma)$, is given by

$$\Sigma^{(\ell)}(\gamma) := \bigcup_{K\subseteq\mathcal{C}^{(\ell)}} \ \bigcup_{T\in\mathcal{I}\left(T^{(\ell)}(K,\gamma)\right)} \sigma(T).$$

The *set of cycles cleared by $TTC(\gamma)$*, denoted by $\Sigma(\gamma)$, is the set of cycles cleared by $TTC(\gamma)$ in some round ℓ,

$$\Sigma(\gamma) := \bigcup_{\ell} \Sigma^{(\ell)}(\gamma).$$

For any cycle $\sigma \in \Sigma(\gamma)$ and time τ we say that the cycle σ is *clearing at time τ* if $\gamma(\tau) \not\leq \underline{x}(\sigma)$ and $\gamma(\tau) \not\geq \overline{x}(\sigma)$. We say that the cycle σ *is cleared at time τ* or *finishes clearing at time τ* if $\gamma^{(l)}(\tau) \leq \underline{x}(\sigma)$ with at least one equality. We remark that for any TTC path γ there may be multiple cycles clearing at a time τ, each corresponding to a different recurrent set. For any TTC path γ the set $\Sigma(\gamma)$ is finite.

Fix two TTC paths γ and γ'. Our goal is to show that they clear the same sets of cycles, $\Sigma(\gamma) = \Sigma(\gamma')$, or equivalently that $\Sigma(\gamma) \cup \Sigma(\gamma') = \Sigma(\gamma) \cap \Sigma(\gamma')$. We will do this by showing that for every cycle $\sigma \in \Sigma(\gamma) \cup \Sigma(\gamma')$, if all cycles in $\Sigma(\gamma) \cup \Sigma(\gamma')$ that block σ are in $\Sigma(\gamma) \cap \Sigma(\gamma')$, then $\sigma \in \Sigma(\gamma) \cap \Sigma(\gamma')$. We first show that this is true in a special case, which can be understood intuitively as the case when the cycle σ appears during the run of $TTC(\gamma)$ and also appears during the run of $TTC(\gamma')$.

Lemma 6. *Let $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ be a continuum economy, and let γ and γ' be two TTC paths for this economy. Let $K \subseteq \mathcal{C}$ and $\underline{t}$ be such that at time $\underline{t}$, γ (γ') has* *available schools C (C'), the paths γ, γ' are at the same point when projected onto the coordinates K, i.e. $\gamma(\underline{t})_K = \gamma'(\underline{t})_K$, and K is a recurrent communication class of $M^C(\gamma(\underline{t}))$ and of $M^{C'}(\gamma'(\underline{t}))$. Suppose that for all schools $c \in K$ and cycles $\sigma' \rhd \sigma$ involving school c, if $\sigma' \in \Sigma(\gamma)$, then σ' is cleared in $TTC(\gamma')$, and vice versa. Suppose also that cycle $\sigma = (K, \underline{x}, \overline{x})$ is cleared in $TTC(\gamma)$, $\gamma(\underline{t}) = \underline{x}$, and at most measure 0 of σ has been cleared by time $\underline{t}$ in $TTC(\gamma')$. Then σ is also cleared in $TTC(\gamma')$.*

Proof. We define the 'interior' of the cycle σ by $X = \{x : \underline{x}_c \le x_c \le \overline{x}_c \forall c \in K, x_{c'} \ge \underline{x}_{c'} \forall c' \notin K\}$. Fix a time u such that $\gamma'(u) \in X$ and let D' denote the set of available schools at time u in $TTC(\gamma')$. Then we claim that K is a recurrent communication class of $M^{D'}(\gamma'(u))$, and that a similar result is true for γ and a similarly defined D. The claim for γ, D follows from the fact that σ is cleared in $TTC(\gamma)$, $\sigma \in \Sigma(\gamma)$. It remains to show that the claim for γ', D' is true. Intuitively, D' is some subset of C', where the schools in the set $C' \setminus D'$ only every point to K and are never pointed to by K and so the cycle remains intact. Formally, by Lemma 4 it suffices to show that $H^{D'}(x)_{K,K}$ is irreducible and $H^{D'}(x)_{K,D'\setminus K}$ is the zero matrix.

We first examine the differences between the matrices $H^{C'}(\gamma'(t))$ and $H^{D'}(\gamma'(u))$. Since K is a recurrent communication class of $M^{C'}(\gamma'(u))$, it holds that $\mathbf{1}\left(H^{C'}(\gamma'(u))_{K,C'\setminus K}\right) = 0$, and so since $K \subseteq D' \subseteq C'$ it follows that $\mathbf{1}\left(H^{D'}(\gamma'(u))_{K,D'\setminus K}\right) = 0$. Moreover, since $\mathbf{1}\left(H^{C'}(\gamma'(u))_{K,C\setminus K}\right) = 0$, all students' top choice schools out of C' or D' are the same (in K), and so $H^{C'}(\gamma'(u))_{K,K} = H^{D'}(\gamma'(u))_{K,K}$ and both matrices are irreducible. Hence K is a recurrent communication class of $M^{D'}(\gamma'(u))$.

We now invoke Theorem 4 to show that in each of the two paths, all the students in the cycle σ clear with each other. In other words, there exists a time τ such that $\gamma(\tau) = \overline{x}_c \forall c \in K$, and similarly there exists a time τ' such that $\gamma'(\tau')_c = \overline{x}'_c \forall c \in K$. Specifically, while the path γ is in the 'interior' of the cycle, that is $\gamma(\tau) \in X$, it follows from Theorem 4 that the projection of the gradient of γ to K is a rescaling of some vector $d^K(\gamma(\tau))$, where $d^K(\cdot)$ depends on $H(\cdot)$ but not on γ. Similarly, while $\gamma'(\tau') \in X$, it holds that the projection of the gradient of γ' to K is a rescaling of the vector $d^K(\gamma'(\tau'))$, for the same function $d^K(\cdot)$. Hence if we let $\pi_K(x)$ denote the projection of a vector x to the coordinates indexed by schools in K, then $\pi_K(\gamma(\gamma^{-1}((\underline{x}, \overline{x}]))) = \pi_K(\gamma'(\gamma'^{-1}((\underline{x}, \overline{x}])))$.

Recall that we have assumed that for all schools $c \in K$ and cycles $\sigma' \rhd \sigma$ involving school c, if $\sigma' \in \Sigma(\gamma)$, then σ' is cleared in $TTC(\gamma')$, and vice versa. This implies that for all $c \in K$, the measure of students assigned to c in time $[0, \underline{t}]$ under $TTC(\gamma)$ is the same as the measure of students assigned to c in time $[0, \underline{t}]$ under $TTC(\gamma')$. Moreover, we have just shown that for any $x \in \gamma(\gamma^{-1}((\underline{x}, \overline{x})))$, $x' \in \gamma'(\gamma'^{-1}((\underline{x}, \overline{x})))$ such that $x_K = x'_K$, if we let $\tau = \gamma^{-1}(x)$ and $\tau' = (\gamma')^{-1}(x')$ then the same measure of students are assigned to c in time $[\underline{t}, \tau]$ under $TTC(\gamma)$ as in time $[\underline{t}, \tau']$ under $TTC(\gamma')$. Since $TTC(\gamma)$ clears σ the moment it exits the interior of σ, this implies that $TTC(\gamma')$ also clears σ the moment it exits the interior. □

We are now ready to prove that the TTC allocation is unique. As the proof takes several steps, we separate it into several smaller claims for readability.

Proof of uniqueness. Let γ and γ' be two TTC paths, and let the sets of cycles associated with $TTC(\gamma)$ and $TTC(\gamma')$ be $\Sigma = \Sigma(\gamma)$ and $\Sigma' = \Sigma(\gamma')$ respectively. We will show that $\Sigma = \Sigma'$.

Let $\sigma = (K, \underline{x}, \overline{x})$ be a cycle in $\Sigma \cup \Sigma'$ such that the following assumption holds:

Assumption 2. *For all $\tilde{\sigma} \rhd \sigma$ it holds that either $\tilde{\sigma}$ is in both Σ and Σ' or $\tilde{\sigma}$ is in neither.*

We show that if σ is in $\Sigma \cup \Sigma'$ then it is in $\Sigma \cap \Sigma'$. Since Σ and Σ' are finite sets, this will be sufficient to show that $\Sigma = \Sigma'$. Without loss of generality we may assume that $\sigma \in \Sigma$.

We give here an overview of the proof. Let $\Sigma_{\rhd\sigma} = \{\tilde{\sigma} \in \Sigma : \tilde{\sigma} \rhd \sigma\}$ denote the set of cycles that are comparable to σ and cleared before σ in $TTC(\gamma)$. Assumption (2) about σ implies that $\Sigma_{\rhd\sigma} \subseteq \Sigma'$. We will show that this implies that no students in σ start clearing under $TTC(\gamma')$ until all the students in σ have the same top available school in $TTC(\gamma')$ as when they clear in $TTC(\gamma)$, or in other words, that if some students in σ start clearing under $TTC(\gamma')$ at time t, then the cycle σ appears at time t. We will then show that once some of the students in σ start clearing under $TTC(\gamma')$ then all of them start clearing. It then follows from Lemma 6 that σ clears under both $TTC(\gamma)$ and $TTC(\gamma')$.

Let ℓ denote the round of $TTC(\gamma)$ in which σ is cleared, $C(x) = \mathcal{C}^{(\ell)}\ \forall x \in \sigma$. We define the times in $TTC(\gamma)$ and $TTC(\gamma')$ when all the cycles in $\Sigma_{\rhd\sigma}$ are cleared, by

$$\begin{aligned}\bar{t}_{\rhd\sigma} &= \min\left\{t : \gamma(t) \le (\underline{\tilde{x}}) \text{ for all } \tilde{\sigma} = \left(\tilde{K}, \underline{\tilde{x}}, (\bar{\tilde{x}})\right) \in \Sigma_{\rhd\sigma} \text{ and } H(\gamma(t)) \ne \mathbf{0}\right\}, \\ \bar{t}'_{\rhd\sigma} &= \min\left\{t : \gamma'(t) \le (\underline{\tilde{x}}) \text{ for all } \tilde{\sigma} = \left(\tilde{K}, \underline{\tilde{x}}, (\bar{\tilde{x}})\right) \in \Sigma_{\rhd\sigma} \text{and } H(\gamma'(t)) \ne \mathbf{0}\right\}.\end{aligned}$$

We define also the times in $TTC(\gamma)$ when σ starts to be cleared and finishes clearing,

$$\underline{t}_\sigma = \max\{t : \gamma(t) \ge \bar{x}\}, \ \bar{t}_\sigma = \min\{t : \gamma(t) \le \underline{x}\}$$

and similarly define the times $\underline{t}'_\sigma = \max\{t : \gamma'(t) \ge \bar{x}\}, \bar{t}_\sigma = \min\{t : \gamma'(t) \le \underline{x}\}$ for $TTC(\gamma')$.

We remark that part of the issue, carried over from the discrete setting, is that these times $\bar{t}_{\rhd\sigma}$ and $\underline{t}_\sigma$ might not match up, and similarly for $\overline{t'}_{\rhd\sigma}$ and $\underline{t'}_\sigma$. In particular, other incomparable cycles could clear at interwoven times. In the continuum model, there may also be sections on the TTC curve at which no school is pointing to a positive density of students. However, all the issues in the continuum case can be addressed using the intuition from the discrete case.

We first show in Claims (1), (2) and (3) that in both $TTC(\gamma)$ and $TTC(\gamma')$, after all the cycles in $\Sigma_{\rhd\sigma}$ are cleared and before σ starts to be cleared, the schools pointed to by students in σ and the students pointed to by schools in K remain constant (up to a set of η-measure 0).

Claim 1. *Let $\sigma = (K, \underline{x}, \bar{x}) \in \Sigma$ satisfy Assumption 2. Suppose there is a school c that some student in σ prefers to all the schools in K. Then school c is unavailable in $TTC(\gamma)$ at any time $t \ge \bar{t}_{\rhd\sigma}$, and unavailable in $TTC(\gamma')$ at any time $t \ge \bar{t}'_{\rhd\sigma}$.*

Proof. Suppose that school c is available in $TTC(\gamma)$ after all the cycles in $\Sigma_{\rhd\sigma}$ are cleared. Then there exists a cycle $\tilde{\sigma}$ clearing at time $\tilde{t} \in \left(\bar{t}_{\rhd\sigma}, \underline{t}_\sigma\right)$ in $TTC(\gamma)$ involving school c. But this means that $\tilde{\sigma} \rhd \sigma$ so $\tilde{\sigma} \in \Sigma_{\rhd\sigma}$, which is a contradiction. Hence the measure of students in $\Sigma_{\rhd\sigma}$ who are assigned to school c is q_c, and the claim follows. □

Claim 2. *In $TTC(\gamma)$, let $\tilde{\Theta}$ denote the set of students cleared in time $\left[\bar{t}_{\rhd\sigma}, \underline{t}_\sigma\right)$ who are preferred by some school in $c \in K$ to the students in σ, that is, θ satisfying $r^\theta_c > \bar{x}_c$. Then $\eta\left(\tilde{\Theta}\right) = 0$.*

Proof. Suppose $\eta\left(\tilde{\Theta}\right) > 0$. Then, since there are a finite number of cycles in $\Sigma(\gamma)$, there exists some cycle $\tilde{\sigma} = \left(\tilde{K}, \underline{\tilde{x}}, (\tilde{\overline{x}})\right) \in \Sigma(\gamma)$ containing a positive η-measure of students in $\tilde{\Theta}$. We show that $\tilde{\sigma}$ is cleared before σ. Since $\tilde{\sigma}$ contains a positive η-measure of students in $\tilde{\Theta}$, it holds that there exist $t_1, t_2 \in \left[\overline{t}_{\rhd\sigma}, \underline{t}_\sigma\right)$ and a school $c \in K$ for which $\underline{\tilde{x}}_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq (\tilde{\overline{x}})_c$. Hence $\overline{x}_c \leq \gamma(\underline{t}_\sigma)_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \tilde{\overline{x}}_c$, so $\tilde{\sigma} \rhd \sigma$ as claimed. But $(\tilde{\underline{x}})_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \gamma\left(\overline{t}_{\rhd\sigma}\right)_c$,so $\tilde{\sigma}$ is not cleared before $\overline{t}_{\rhd\sigma}$, contradicting the definition of $\overline{t}_{\rhd\sigma}$. □

Claim 3. *In $TTC(\gamma')$, let $\tilde{\Theta}$ denote the set of students cleared in time $\left[\overline{t}'_{\rhd\sigma}, \underline{t}'_\sigma\right)$ who are preferred by some school in $c \in K$ to the students in σ, that is, θ satisfying $r^\theta_c > \overline{x}_c$. Then $\eta\left(\tilde{\Theta}\right) = 0$.*

Proof. Suppose $\eta\left(\tilde{\Theta}\right) > 0$. Then, since there are a finite number of cycles in $\Sigma(\gamma')$, there exists some cycle $\tilde{\sigma} = \left(\tilde{K}, \underline{\tilde{x}}, (\tilde{\overline{x}})\right) \in \Sigma(\gamma')$ containing a positive η-measure of students in $\tilde{\Theta}$. We show that $\tilde{\sigma}$ is cleared before σ. Since $\tilde{\sigma}$ contains a positive η-measure of students in $\tilde{\Theta}$, it holds that there exist $t_1, t_2 \in \left[\overline{t}'_{\rhd\sigma}, \underline{t}'_\sigma\right)$ for which $\underline{\tilde{x}}_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq (\tilde{\overline{x}})_c$. Hence $\overline{x}_c \leq \gamma'(\underline{t}'_\sigma)_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \tilde{\overline{x}}_c$,so $\tilde{\sigma} \rhd \sigma$ and must be cleared before σ. Moreover, $(\tilde{\underline{x}})_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \gamma\left(\overline{t}'_{\rhd\sigma}\right)_c$, so it follows from the definition of $\overline{t}'_{\rhd\sigma}$ that $\tilde{\sigma} \notin \Sigma_{\rhd\sigma}$, but since we assumed that $\tilde{\sigma} \in \Sigma'$ it follows that $\tilde{\sigma} \in \Sigma' \setminus \Sigma$, contradicting assumption (2) on σ. □

We now show in Claims (4) and (5) that in both $TTC(\gamma)$ and $TTC(\gamma')$ the cycle σ starts clearing when students in the cycle σ start clearing. We formalize this in the continuum model by considering the coordinates of the paths γ, γ' at the time $\underline{t}_\sigma$ when the cycle σ starts clearing, and showing that, for all coordinates indexed by schools in K, this is equal to $\overline{x}$.

Claim 4. $\pi_K(\gamma(\underline{t}_\sigma)) = \pi_K(\overline{x})$.

Proof. The definition of $\underline{t}_\sigma$ implies that $\gamma(\underline{t}_\sigma)_c \geq \overline{x}_c$ for all $c \in K$. Suppose there exists $c \in K$ such that $\gamma(\underline{t}_\sigma)_c > \overline{x}_c$. Since σ starts clearing at time $\underline{t}_\sigma$, for all $\varepsilon > 0$ school c must point to a non-zero measure of students in σ over the time period $[\underline{t}_\sigma, \underline{t}_\sigma + \varepsilon]$, whose scores r^θ_c satisfy $\gamma(\underline{t}_\sigma)_c \geq r^\theta_c \geq \gamma(\underline{t}_\sigma + \varepsilon)_c$. For sufficiently small ε the continuity of $\gamma(\cdot)$ and the assumption that $\gamma(\underline{t}_\sigma)_c > \overline{x}_c$ implies that $r^\theta_c \geq \gamma(\underline{t}_\sigma + \varepsilon)_c > \overline{x}_c$, which contradicts the definition of $\overline{x}_c$. □

Claim 5. $\pi_K(\gamma'(\underline{t}'_\sigma)) = \pi_K(\overline{x})$.

As in the proof of Claim (4), the definition of $\underline{t}'_\sigma$ implies that $\gamma'(\underline{t}'_\sigma)_c \geq \overline{x}_c = \gamma(\underline{t}_\sigma)_c$ for all $c \in K$. Since we cannot assume that σ is the cycle that is being cleared at time $\underline{t}'_\sigma$ in $TTC(\gamma')$, the proof of Claim (5) is more complicated than that of the Claim (4) and takes several steps.

We rely on the fact that K is a recurrent communication class in $TTC(\gamma)$, and that all cycles comparable to σ are already cleared in $TTC(\gamma')$. The underlying concept is very simple in the discrete model, but is complicated in the continuum by the definition of the TTC path in terms of specific points, as opposed to measures of students, and the need to account for sets of students of η-measure 0.

Let $K_=$ be the set of coordinates in K at which equality holds, $\gamma'(\underline{t}'_\sigma)_c = \gamma(\underline{t}_\sigma)_c$, and let $K_>$ be the set of coordinates in K where strict inequality holds, $\gamma'(\underline{t}'_\sigma)_c > \gamma(\underline{t}_\sigma)_c$. It suffices to show that $K_>$ is empty. We do this by showing that under $TTC(\gamma')$ at time $\underline{t}'_\sigma$, every school in $K_>$ points to a zero density of students, and some school in $K_=$ points to a non-zero density of students, and so if both sets are non-empty this contradicts the marginal trade balance equations. In what follows, let C denote the set of available schools in $TTC(\gamma)$ at time $\underline{t}_\sigma$, and let C' denote the set of available schools in $TTC(\gamma')$ at time $\underline{t}'_\sigma$.

Claim 6. *Suppose that $c \in K_>$. Then there exists $\varepsilon > 0$ such that in $TTC(\gamma')$, the set of students pointed to by school c in time $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ has η-measure 0, i.e. $\widetilde{H}^{C'}(\gamma'(\underline{t}'_\sigma))_{cb} = 0$.*

Proof. Since $c \in K_>$ it holds that $\gamma'(\underline{t}'_\sigma)_c > \overline{x}_c$, and since γ' is continuous, for sufficiently small ε it holds that $\gamma'(\underline{t}'_\sigma + \varepsilon)_c > \overline{x}_c$. Hence the set of students that school c points to in time $[\underline{t}'_\sigma, \underline{t}'_\sigma + \varepsilon]$ is a subset of those with score r^θ_c satisfying $\gamma'(\underline{t}'_\sigma)_c \geq r^\theta_c \geq \gamma'(\underline{t}'_\sigma + \varepsilon)_c > \overline{x}_c$. By assumption (2) and Claim (3) any cycle $\tilde{\sigma}$ clearing some of these students contains at most measure 0 of them, since $\tilde{\sigma}$ is cleared after $\Sigma_{\triangleright\sigma}$ and before σ. Since there is a finite number of such cycles the set of students has η-measure 0. □

Claim 7. *If $c \in K_=$, $b \in K$ and $\widetilde{H}^{C}(\gamma(\underline{t}_\sigma))_{cb} > 0$, then $\widetilde{H}^{C'}(\gamma'(\underline{t}'_\sigma))_{cb} > 0$.*

Proof. Since every $\widetilde{H}^{C}(\gamma'(\underline{t}'_\sigma))_{cb}$ is a positive multiple of $H^{b|C}_c(\gamma'(\underline{t}'_\sigma))$, it suffices to show that $H^{b|C'}_c(\gamma'(\underline{t}'_\sigma)) > 0$. Let $\Sigma'_-(\varepsilon) \overset{def}{=} (\gamma'(\underline{t}'_\sigma) - \varepsilon \cdot e_c, \gamma'(\underline{t}'_\sigma)]$. We first show that for sufficiently small ε it holds that $\eta^{b|C}\left(\Sigma'_-(\varepsilon)\right) = \Omega(\varepsilon)$. Let $\Sigma_-(\varepsilon) \overset{def}{=}$

$(\gamma(\underline{t}_\sigma) - \varepsilon \cdot e^c, \gamma(\underline{t}_\sigma)]$. Since $\widetilde{H}^C(\gamma(\underline{t}_\sigma))_{cb} > 0$, it follows from the definition of $H_c^{b|C}(\cdot)$ that $H_c^{b|C}(x) \doteq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta^{b|C}(\Sigma_-(\varepsilon)) > 0$ and hence $\eta^{b|C}(\Sigma_-(\varepsilon)) = \Omega(\varepsilon)$ for sufficiently small ε. Moreover, at most η-measure 0 of the students in $\Sigma_-(\varepsilon)$ are not in the cycle σ. Finally, $\Sigma'_-(\varepsilon) \supseteq \Sigma_-(\varepsilon) \setminus \Sigma_+(\varepsilon)$, where $\Sigma_+(\varepsilon) \overset{def}{=} (\gamma(\underline{t}_\sigma) + \varepsilon \cdot e_c, \gamma(\underline{t}_\sigma)]$. If $\varepsilon < \overline{x}_c - \underline{x}_c$ then η-measure 0 of the students in $\Sigma_+(\varepsilon)$ are not cleared by cycle σ. Hence $\eta^{b|C}\left(\Sigma'_-(\varepsilon)\right) \geq \eta^{b|C}(\Sigma_-(\varepsilon)) - \eta^{b|C}(\Sigma_+(\varepsilon)) = \Omega(\varepsilon)$.

Suppose for the sake of contradiction that $H_c^{b|C'}(\gamma'(\underline{t}'_\sigma)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta^{b|C'}\left(\Sigma'_-(\varepsilon)\right) = 0$, so that $\eta^{b|C'}\left(\Sigma'_-(\varepsilon)\right) = o(\varepsilon)$ for sufficiently small ε. Then there is a school $b' \neq b$ and type $\theta \in \Theta^{b|C} \cap \Theta^{b'|C'}$ such that there is an η-measure $\Omega(\varepsilon)$ of students in σ with type θ. Since $b' \in C'$ it is available in $TTC(\gamma')$ at time $\underline{t}'_\sigma$, and by Claim (1) it holds that $b' \in K$. Moreover, $\theta \in \Theta^{b|C}$ implies that θ prefers school b to all other schools in K, so $b = b'$, contradiction. □

Proof of Claim (5). Suppose for the sake of contradiction that $K_>$ is nonempty. Since some students in σ are being cleared in $TTC(\gamma')$ at time $\underline{t}'_\sigma$, by Claim (3) there exists $c \in K = K_= \cup K_>$ and $b \in K$ such that $\widetilde{H}^{C'}(\gamma'(\underline{t}'_\sigma))_{cb} > 0$. If $c \in K_>$ this contradicts Claim (6). If $c \in K_=$, then $\widetilde{H}^{C'}(\gamma(\underline{t}_\sigma))_{cb} > 0$ and so by Claim (1) $\widetilde{H}^C(\gamma(\underline{t}_\sigma))_{cb} > 0$. Moreover, $K = K_= \cup K_>$ is a recurrent communication class of $\widetilde{H}^C(\gamma(\underline{t}_\sigma))$, so there exists a chain $c = c_0 - c_1 - c_2 - \cdots - c_n$ such that $\widetilde{H}^C(\gamma(\underline{t}_\sigma))_{c_i c_{i+1}} > 0$ for all $i < n$, $c_i \in K_=$ for all $i < n-1$, and $c_{n-1} \in K_>$. By Claim (7) $\widetilde{H}^{C'}(\gamma'(\underline{t}'_\sigma))_{c_i c_{i+1}} > 0$ for all $i < n$. But since $c_{n-1} \in K_>$, by Claim (6) $\widetilde{H}^{C'}(\gamma(\underline{t}'_\sigma))_{c_{n-1} c_n} = 0$, which gives the required contradiction. □

Proof that $\Sigma = \Sigma'$. We have shown in Claims (4) and (5) that for our chosen $\sigma = (K, \underline{x}, \overline{x})$, it holds that $\gamma(\underline{t}_\sigma)_K = \gamma'(\underline{t}'_\sigma)_K = \underline{x}_K$. Invoking Claims (2) and (3) and Lemma 6 shows that σ is cleared under both $TTC(\gamma)$ and $TTC(\gamma')$. Hence $\Sigma = \Sigma'$, as required. □

D.4 Proof of Proposition 2

In this section, we show that given a discrete economy, the cutoffs of TTC in a continuum embedding Φ give the same assignment as TTC on the discrete model,

$$\mu_{dTTC}(s \mid E) = \max_{\succ^s} \{c : r_b^s \geq p_b^c \text{ for some } b\} = \mu_{cTTC}(\theta^s \mid \Phi(E)) \;\; \forall \theta^s \in I^s.$$

The intuition behind this result is that TTC is essentially performing the same assignments in both models, with discrete TTC assigning students to schools in discrete steps, and continuum TTC assigning students to schools continuously, in fractional amounts. By considering the progression of continuum TTC at the discrete time steps when individual students are fully assigned, we obtain the same outcome as discrete TTC.

Proof. For a discrete economy $E = (\mathcal{C}, \mathcal{S}, \succ_{\mathcal{C}}, \succ^{S}, q)$ with $N = |\mathcal{S}|$ students, we define the continuum economy $\Phi(E) = (\mathcal{C}, \Theta, \eta, \frac{q}{N})$ as follows. For each student $s \in \mathcal{S}$ and school $c \in \mathcal{C}$, recall that $r_c^s = |\{s' \mid s \succ_c s'\}| / |\mathcal{S}|$ is the percentile rank of s at c. We identify each student $s \in \mathcal{S}$ with the N-dimensional cube $I^s = \succ^s \times \prod_{c \in \mathcal{C}} \left[r_c^s, r_c^s + \frac{1}{N}\right)$ of student types, and define η to have constant density $\frac{1}{N} \cdot N^N$ on $\cup_s I^s$ and 0 everywhere else.

We construct a discrete cycle selection rule ψ and TTC path γ such that TTC on the discrete economy E with cycle selection rule ψ gives the same allocation as $TTC(\gamma)$. Since the assignment of discrete TTC is unique (Shapley and Scarf, 1974), and the assignment in the continuum model is unique (Proposition 2), this proves the theorem.

The discrete cycle selection rule ψ is defined by taking all available cycles in the pointing graph obtained by having students point to their favorite school, and schools to their favorite student. The TTC path γ is defined by taking valid directions $\boldsymbol{d}(x)$ that essentially use all available cycles in the pointing graph. Formally, at each point x, let C be the set of available schools, let $K(x)$ be the set of all students in recurrent communication classes of $\widetilde{H}(x)$, and let $d_c(x) = \frac{1}{|K(x)|}$ if $c \in K$ and 0 otherwise.

Let X be the set of points x such that x_c is a multiple of $\frac{1}{N}$ for all $c \notin K(x)$; we will show that the TTC path stays within this set of points. Note that for each x the matrix $N \times \widetilde{H}(x)$ is the adjacency matrix of the pointing graph (where school b points to school c if some student pointed to by b wants c), and so $\boldsymbol{d}(x) = \boldsymbol{d}(x) \cdot \widetilde{H}(x)$ for all $x \in X$. Now consider the TTC path γ satisfying $\gamma'(t) = d(\gamma(t))$. The path starts at $\gamma(0) = \mathbf{1} \in X$. Moreover, at any time t, if $\gamma(t) \in X$ then $\gamma'(t) = \boldsymbol{d}(\gamma(t))$ points along the diagonal in the projection onto the coordinates K, and is 0 along all other coordinates. Hence $\gamma(t) \in X$ for all t.

We now show that by considering the progression of continuum TTC at the discrete time steps when individual students are fully assigned, we obtain the same cycles and outcome as discrete TTC. Let $t_1, t_2, \ldots$ be the discrete set of times when a student s is first fully assigned, that is $\{t_i\} = \cup_s \left\{\inf \left\{t \mid \exists c \in C \text{ s.t. } \gamma_c(t) \le r_c^\theta \ \forall \theta \in I^s\right\}\right\}$.

For every two students s, s' and school c it holds that the projections I_c^s and $I_c^{s'}$ of I^s and $I^{s'}$ onto the cth coordinates are non-overlapping, i.e. either for all $\theta \in I^s, \theta' \in I^{s'}$ it holds that $r_c^\theta < r_c^{\theta'}$, or for all $\theta \in I^s, \theta' \in I^{s'}$ it holds that $r_c^\theta > {r^{\theta'}}_c$. Since all the capacities are multiples of $\frac{1}{N}$, it follows that $\gamma_c(t_i)$ is a multiple of $\frac{1}{N}$ for all c, i and schools fill at a subset of the set of times $\{t_i\}$.

In other words, we have shown that for every i, if S is the set students who are allocated a seat at time t_i, then $S \cup \mu(S)$ are the agents in the maximal union of cycles in the pointing graph at time t_{i-1}. Hence γ finishes clearing the cubes corresponding to the same set of cycles at t_i as ψ clears in step i. It follows that $\mu_{dTTC}(s \mid E) = \mu_{cTTC}(\theta^s \mid \Phi(E))$, and by definition it holds that $\mu_{cTTC}(\theta^s \mid \Phi(E)) = \max_{\succ^s}\{c : r_b^s \geq p_b^c \text{ for some } b\}$. □

D.5 Proof of Theorem 3

To prove Theorem 3, we will want some way of comparing two TTC paths γ and $\tilde{\gamma}$ obtained under two continuum economies differing only in their measures η and $\tilde{\eta}$.

Definition 7. Let γ and $\tilde{\gamma}$ be increasing continuous functions from $[0,1]$ to $[0,1]^{\mathcal{C}}$ with $\gamma(0) = \tilde{\gamma}(0)$. We say that $\gamma(\tau)$ *dominates* $\tilde{\gamma}(\tau)$ *via school* c if

$$\begin{aligned} \gamma_c(\tau) &= \tilde{\gamma}_c(\tau), \text{ and} \\ \gamma_b(\tau) &\leq \tilde{\gamma}_b(\tau) \text{ for all } b \in \mathcal{C}. \end{aligned}$$

We remark that, somewhat unintuitively, the condition $\gamma(\tau) \leq \tilde{\gamma}(\tau)$ implies that more students are offered seats under γ than γ', since higher ranks give more restrictive sets. We also say that γ *dominates* $\tilde{\gamma}$ *via school* c *at time* τ. If γ and γ' are TTC paths, we can interpret this as school c being more demanded under γ, since with the same rank at c, in γ students are competitive with more ranks at other schools b.

We now show that any two non-increasing continuous paths γ, γ' starting and ending at the same point can be re-parametrized so that for all t there exists a school $c(\tau)$ such that γ dominates γ' via school $c(\tau)$ at time t. We first show that, if $\gamma(0) \leq \tilde{\gamma}(0)$, then there exists a re-parametrization of γ such that γ dominates γ' on some interval starting at 0.

Lemma 7. *Suppose γ, $\tilde{\gamma}$ are a pair of non-increasing functions $[0,1] \to [0,1]^{\mathcal{C}}$ such that $\gamma(0) \leq \tilde{\gamma}(0)$. Then there exist coordinates c, b, a time $\bar{t}$ and an increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $\gamma_b\left(g\left(\bar{t}\right)\right) = \tilde{\gamma}_b\left(\bar{t}\right)$, and for all $\tau \in \left[0, \bar{t}\right]$ it holds that*

$$\gamma_c\left(g\left(\tau\right)\right) = \tilde{\gamma}_c\left(\tau\right) \text{ and } \gamma\left(g\left(\tau\right)\right) \leq \tilde{\gamma}\left(\tau\right).$$

That is, if we renormalize the time parameter τ of $\gamma(\tau)$ so that γ and $\tilde{\gamma}$ agree along the cth coordinate, then γ dominates $\tilde{\gamma}$ via school c at all times $\tau \in \left[0, \bar{t}\right]$, and also dominates via school b at time $\bar{t}$.

Proof. The idea is that if we take the smallest function g such that $\gamma_c\left(g\left(\tau\right)\right) = \tilde{\gamma}_c\left(\tau\right)$ for some coordinate c and all τ sufficiently small, then $\gamma\left(g\left(\tau\right)\right) \leq \tilde{\gamma}\left(\tau\right)$ for all τ sufficiently small. The lemma then follows from continuity. We make this precise.

Fix a coordinate c. Let $g^{(c)}$ be the renormalization of γ so that γ and $\tilde{\gamma}$ agree along the cth coordinate, i.e. $\gamma_c\left(g^{(c)}\left(\tau\right)\right) = \tilde{\gamma}_c\left(\tau\right)$ for all τ.

For all τ, we define the set $\kappa_{>}^{(c)}\left(\tau\right) = \left\{b \mid \gamma_b\left(g^{(c)}\left(\tau\right)\right) > \tilde{\gamma}_b\left(\tau\right)\right\}$ of schools b along which the γ curve renormalized along coordinate c has larger b-value at time τ than $\tilde{\gamma}_b$ has at time τ, and similarly define the set $\kappa_{=}^{(c)}\left(\tau\right) = \left\{b \mid \gamma_b\left(g^{(c)}\left(\tau\right)\right) = \tilde{\gamma}_b\left(\tau\right)\right\}$ where the renormalized γ curve is equal to $\tilde{\gamma}$. It suffices to show that there exists b, c and a time $\bar{t}$ such that $\kappa_{>}^{(c)}\left(\tau\right) = \emptyset$ for all $\tau \in \left[0, \bar{t}\right]$ and $b \in \kappa_{=}^{(c)}\left(\bar{t}\right)$.

Since γ and $\tilde{\gamma}$ are continuous, there exists some maximal $\bar{t}^{(c)} > 0$ such that the functions $\kappa_{>}^{(c)}\left(\cdot\right)$ and $\kappa_{=}^{(c)}\left(\cdot\right)$ are constant over the interval $\left(0, \bar{t}^{(c)}\right)$. If there exists c such that $\kappa_{>}^{(c)}\left(\tau\right) = \emptyset$ for all $\tau \in \left(0, \bar{t}^{(c)}\right)$ then by continuity there exists some time $\bar{t} \leq \bar{t}^{(c)}$ and school b such that $b \in \kappa_{=}^{(c)}\left(\bar{t}\right)$ and we are done. Hence we may assume that for all c it holds that $\kappa_{>}^{(c)}\left(\tau\right) = C_{>}^{(c)}$ for all $\tau \in \left(0, \bar{t}^{(c)}\right)$ for some fixed non-empty set $C_{>}^{(c)}$. We will show that this leads to a contradiction.

We first claim that if $b \in C_{>}^{(c)}$, then $g^{(b)}\left(\tau\right) > g^{(c)}\left(\tau\right)$ for all $\tau \in \left(0, \bar{t}\right)$. This is because γ is increasing and $\gamma_b\left(g^{(b)}\left(\tau\right)\right) = \tilde{\gamma}_b\left(\tau\right) > \gamma_b\left(g^{(c)}\left(\tau\right)\right)$ for all $\tau \in \left(0, \bar{t}\right)$, where the equality follows from the definition of $g^{(b)}$ and the inequality since $b \in C_{>}^{(c)}$. But this completes the proof, since it implies that for all c there exists b such that $g^{(b)}\left(\tau\right) > g^{(c)}\left(\tau\right)$ for all $\tau \in \left(0, \bar{t}\right)$, which is impossible since there are a finite number of schools $c \in \mathcal{C}$. □

We are now ready to show that there exists a re-parametrization of γ such that γ always dominates $\tilde{\gamma}$ via some school.

Lemma 8. *Suppose $\bar{t} \geq 0$ and γ, $\tilde{\gamma}$ are a pair of non-increasing functions $\left[0, \bar{t}\right] \rightarrow [0,1]^{\mathcal{C}}$ such that $\gamma(0) \leq \tilde{\gamma}(0) = 1$ with equality on at least one coordinate, and $0 = \gamma(1) \leq \tilde{\gamma}(1)$ with equality on at least one coordinate. Then there exists an increasing function $g : \left[0, \bar{t}\right] \rightarrow \mathbb{R}$ such that for all $\tau \geq 0$, there exists a school $c(\tau)$ such that $\gamma(g(\tau))$ dominates $\tilde{\gamma}(\tau)$ via school $c(\tau)$.*

Proof. Without loss of generality let us assume that $\bar{t} = 1$. Fix a coordinate c. We define $g^{(c)}$ to be the renormalization of γ so that γ and $\tilde{\gamma}$ agree along the cth coordinate. Formally, let $\underline{t}^{(c)} = \min\{\tau \mid \gamma_c(0) \geq \tilde{\gamma}_c(\tau)\}$ and define $g^{(c)}$ so that $\gamma_c\left(g^{(c)}(\tau)\right) = \tilde{\gamma}_c(\tau)$ for all $\tau \in \left[\underline{t}^{(c)}, 1\right]$. Let $A^{(c)}$ be the set of times τ such that $\gamma\left(g^{(c)}(\tau)\right)$ dominates $\tilde{\gamma}(\tau)$. The idea is to pick g to be equal to $g^{(c)}$ in $A^{(c)}$. In order to do this formally, we need to show that the sets $A^{(c)}$ cover $[0,1]$, and then turn (a suitable subset of) $A^{(c)}$ into a union of disjoint closed intervals, on each of which we can define g$(\cdot) \equiv g^{(c)}(\cdot)$.

We first show that $\cup_c A^{(c)} = [0,1]$. Suppose not, so there exists some time τ such that for all c such that $\tau \geq \underline{t}^{(c)}$ there exists b such that $\gamma_b\left(g^{(c)}(\tau)\right) > \tilde{\gamma}_b(\tau)$. This implies that $\tilde{\gamma}_b(\tau) \leq \gamma_b(0)$, and so there exists $g^{(b)}(\tau)$ such that $\tilde{\gamma}_b(\tau) = \gamma_b\left(g^{(b)}(\tau)\right)$. Since γ is increasing, this implies that for all c such that $\tau \geq \underline{t}^{(c)}$ there exists b such that $g^{(c)}(\tau) < g^{(b)}(\tau)$, which is a contradiction since the set of such schools is finite but non-empty (since $\gamma(0) \leq \tilde{\gamma}(0) = 1$, with equality on at least one coordinate).

We now turn (a suitable subset of $A^{(c)}$) into a union of disjoint closed intervals. By continuity, $A^{(c)}$ is closed. Consider the closure of the interior of $A^{(c)}$, which we denote by $B^{(c)}$. Since the interior of $A^{(c)}$ is open, it is a countable union of open intervals, and hence $B^{(c)}$ is a countable union of disjoint closed intervals. To show that $\cup_{c\in\mathcal{C}} B^{(c)} = [0,1]$, fix a time $\tau \in [0,1]$. As $\cup_c A^{(c)} = [0,1]$, there exists c such that $\gamma\left(g^{(c)}(\tau)\right) \leq \tilde{\gamma}(\tau)$. Hence we may invoke Lemma 7 to show that there exists some school b, time $\overline{\tau} > \tau$ and an increasing function g such that $\gamma_b\left(g\left(g^{(c)}(\tau')\right)\right) = \tilde{\gamma}_b(\tau')$ and $\gamma\left(g\left(g^{(c)}(\tau')\right)\right) \leq \tilde{\gamma}(\tau')$ for all $\tau' \in [\tau, \overline{\tau}]$. But by the definition of $g^{(b)}(\cdot)$ this means that $\gamma_b\left(g\left(g^{(c)}(\tau')\right)\right) = \tilde{\gamma}_b(\tau') = \gamma_b\left(g^{(b)}(\tau')\right)$ for all $\tau' \in [\tau, \overline{\tau}]$, and so $g \circ g^{(c)} = g^{(b)}$ and we have shown that $[\tau, \overline{\tau}] \subseteq B^{(b)}$. Hence we may write $[0,1] = \cup_n\{T_n\}$ as a countable union of closed intervals such that any pair of intervals intersects at most at their endpoints, and each interval T_n is a subset of $B^{(c)}$ for some c. For each T_n fix some $c(n) = c$ so that $T_n \subseteq B^{(c)}$. Intuitively, this means that at any time $\tau \in T_n$ it holds that $\gamma\left(g^{(c(n))}(\tau)\right)$ dominates $\tilde{\gamma}(\tau)$ via school $c(n)$.

We now construct a function g that satisfies the required properties as follows. If $\tau \in T_n \subseteq B^{(i)}$, let $g(\tau) = g^{(c)}(\tau)$. Now g is well-defined despite the possibility that $T_n \cap T_m \neq \emptyset$. This is because if τ is in two different intervals T_n, T_m, then $\gamma_{c(n)}\left(g^{(c(n))}(\tau)\right) = \tilde{\gamma}_{c(n)}(\tau) \geq \gamma_{c(n)}\left(g^{(c(m))}(\tau)\right)$ (by domination via $c(n)$ and $c(m)$ respectively), and $\gamma_{c(m)}\left(g^{(c(m))}(\tau)\right) = \tilde{\gamma}_{c(m)}(\tau) \geq \gamma_{c(m)}\left(g^{(c(n))}(\tau)\right)$ (by domination via $c(m)$ and $c(n)$ respectively), and so $g^{(c(n))}(\tau) \leq g^{(c(m))}(\tau) \leq g^{(c(n))}(\tau)$ and we can pick one value for g that satisfies all required properties. Now by definition $\gamma(g(\tau))$ dominates $\tilde{\gamma}(\tau)$ via school $c(\tau) = c(n)$, and moreover g is defined on all of $[0,1]$ since $\cup_{c\in\mathcal{C}} B^{(c)} = [0,1]$. This completes the proof. □

Consider two continuum economies $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ and $\tilde{\mathcal{E}} = (\mathcal{C}, \Theta, \tilde{\eta}, q)$, where the measures η and $\tilde{\eta}$ satisfy the assumptions given in Section 3. Suppose also that the measure η and $\tilde{\eta}$ have total variation distance ε and have full support. Let γ be a TTC path for economy $\mathcal{E}$, and let $\tilde{\gamma}$ be a TTC path for economy $\tilde{\mathcal{E}}$. Consider any school c and any points $x \in Im(\gamma)$, $\tilde{x} \in Im(\tilde{\gamma})$ such that $x_c = \tilde{x}_c$, and both are cleared in the first round of their respective TTC runs, $C(x|\gamma) = \mathcal{C}$, $C(c|\tilde{\gamma}) = \mathcal{C}$. We show that the set of students allocated to school c when running $TTC(\gamma)$ up to x differs from the set of students allocated to school c when running $TTC(\tilde{\gamma})$ up to $\tilde{x}$ by a set of measure $O(\varepsilon|\mathcal{C}|)$.

Proposition 11. *Suppose that γ, $\tilde{\gamma}$ are TTC paths in one round of the continuum economies $\mathcal{E}$ and $\tilde{\mathcal{E}}$ respectively, where the set of available schools C is the same in these rounds of $TTC(\gamma)$ and $TTC(\gamma')$. Suppose also that γ starts and ends at x, y and $\tilde{\gamma}$ starts and ends at $\tilde{x}, \tilde{y}$, where there exist $b, c \in C$ such that $x_b = \tilde{x}_b$, $y_c = \tilde{y}_c$, and $x_a \leq \tilde{x}_a$, $y_a \leq \tilde{y}_a$ for all $a \in \mathcal{C}$. Then for all $c \in C$, the set of students with ranks in $(y, x]$ under $\mathcal{E}$ and ranks in $(\tilde{y}, \tilde{x}]$ under $\tilde{\mathcal{E}}$ who are assigned to c under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ has measure $O(\varepsilon|C|)$.*[38]

Proof. By Lemma 8, we may assume without loss of generality that γ and $\tilde{\gamma}$ are parametrized such that $x = \gamma(0), y = \gamma(1)$ and $\tilde{x} = \tilde{\gamma}(0), \tilde{y} = \tilde{\gamma}(1)$, and for all times $\tau \leq 1$ there exists a school $c(\tau)$ such that $\gamma(\tau)$ dominates $\tilde{\gamma}(\tau)$ via school $c(\tau)$.

Let $T_c = \{\tau \leq 1 : c(\tau) = c\}$ be the times when γ dominates $\tilde{\gamma}$ via school c. We remark that, by our construction in Lemma 8, we may assume that T_c is the countable union of disjoint closed intervals, and that if $c \neq c'$ then T_c and $T_{c'}$ have disjoint interiors.

[38]This is according to both measures η and $\tilde{\eta}$.

Since γ is a TTC path for $\mathcal{E}$ and $\tilde{\gamma}$ is a TTC path for $\tilde{\mathcal{E}}$, by integrating over the marginal trade balance equations we can show that the following trade balance equations hold,

$$\eta\left(\mathcal{T}_c\left(\gamma; T_c\right)\right) = \eta\left(\mathcal{T}^{c|C}\left(\gamma; T_c\right)\right) \text{ for all } c \in C. \tag{4}$$
$$\tilde{\eta}\left(\mathcal{T}_c\left(\tilde{\gamma}; T_c\right)\right) = \tilde{\eta}\left(\mathcal{T}^{c|C}\left(\tilde{\gamma}; T_c\right)\right) \text{ for all } c \in C. \tag{5}$$

Since γ dominates $\tilde{\gamma}$ via school b at all times $\tau \in T_b$, we have that

$$\mathcal{T}_b\left(\gamma; T_b\right) \subseteq \mathcal{T}_b\left(\tilde{\gamma}; T_b\right). \tag{6}$$

Moreover, by the choice of parametrization, $\cup_b T_b = [0, 1]$ and so, since $x \leq \tilde{x}$,

$$\cup_{b,c}\mathcal{T}^{c|C}\left(\gamma; T_b\right) \supseteq \cup_{b,c}\mathcal{T}^{c|C}\left(\tilde{\gamma}; T_b\right). \tag{7}$$

Now since $\eta, \tilde{\eta}$ have total variation ε, for every school c it holds that

$$\begin{aligned} \eta\left(\mathcal{T}^{c|C}\left(\gamma; T_c\right) \setminus \mathcal{T}^{c|C}\left(\tilde{\gamma}; T_c\right)\right) &\leq \eta\left(\mathcal{T}^{c|C}\left(\gamma; T_c\right)\right) - \eta\left(\mathcal{T}^{c|C}\left(\tilde{\gamma}; T_c\right)\right) + \varepsilon \text{ (by (7))} \\ &= \eta\left(\mathcal{T}_c\left(\gamma; T_c\right)\right) - \tilde{\eta}\left(\mathcal{T}_c\left(\tilde{\gamma}; T_c\right)\right) + \varepsilon \text{ (by (4) and (5))} \\ &\leq 2\varepsilon \text{ (by (6))}, \end{aligned} \tag{8}$$

Also, for all schools $b \neq c$, since η has full support, it holds that

$$\eta\left(\mathcal{T}^{c|C}\left(\gamma; T_b\right) \setminus \mathcal{T}^{c|C}\left(\tilde{\gamma}; T_b\right)\right) \leq \frac{M}{m}\eta\left(\mathcal{T}^{b|C}\left(\gamma; T_b\right) \setminus \mathcal{T}^{b|C}\left(\tilde{\gamma}; T_b\right)\right). \tag{9}$$

Hence, as T_b have disjoint interiors,

$$\begin{aligned} \eta\left(\mathcal{T}^{c|C}\left(\gamma; 1\right) \setminus \mathcal{T}^{c|C}\left(\tilde{\gamma; 1}\right)\right) &= \sum_{b \in C}\left(\eta(\mathcal{T}^{c|C}\left(\gamma; T_b\right)) - \eta(\mathcal{T}^{c|C}\left(\tilde{\gamma}; T_b\right)\right) \text{ (by (7))} \\ &\leq \sum_{b \in C}\eta\left(\mathcal{T}^{c|C}\left(\gamma; T_b\right) \setminus \mathcal{T}^{c|C}\left(\tilde{\gamma}; T_b\right)\right) \\ &\leq \sum_{b \in C}\frac{M}{m}\eta\left(\mathcal{T}^{b|C}\left(\gamma; T_b\right) \setminus \mathcal{T}^{b|C}\left(\tilde{\gamma}; T_b\right)\right) \text{ (by (9))} \\ &\leq 2|C|\varepsilon\frac{M}{m} \text{ (by (8))}. \end{aligned}$$

That is, given a school c, the set of students assigned to school c with score $r^\theta \nleq x$ under γ and not assigned to school c with score $r^\theta \nleq \tilde{x}$ under $\tilde{\gamma}$ has η-measure $O(\varepsilon |C|)$. The result for $\tilde{\eta}$ follows from the fact that the total variation distance of η and $\tilde{\eta}$ is ε. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Assume without loss of generality that the schools are indexed such that the stopping times $t^{(c)}$ for $TTC(\gamma)$ satisfy $t^{(1)} \leq t^{(2)} \leq \cdots \leq t^{(|\mathcal{C}|)}$ i.e. school c_ℓ is the ℓth school to reach capacity under $TTC(\gamma)$. Let σ be a permutation such that the stopping times $\tilde{t}^{(c)}$ for $TTC(\tilde{\gamma})$ satisfy $t^{(\sigma(1))} \leq \cdots \leq t^{(\sigma(|\mathcal{C}|))}$. We show by induction on ℓ that $\sigma(\ell) = \ell$ and that for all schools c, the set of students assigned to c under $TTC(\gamma)$ by time $t^{(\ell)}$ and not under $TTC(\tilde{\gamma})$ by time $\tilde{t}^{(\sigma(\ell))}$ has η-measure $O(\varepsilon\ell |\mathcal{C}|)$. This will prove the theorem.

We first consider the base case $\ell = 1$. Let $x = \tilde{x} = \gamma(0)$ and $y = \gamma\left(t^{(1)}\right)$. Define $\tilde{y} \in Im(\tilde{\gamma})$ to be the minimal point such that $y \leq \tilde{y}$ and there exists c such that $y_c = \tilde{y}_c$. We show that $\tilde{y}$ is near $\tilde{\gamma}\left(\tilde{t}^{(1)}\right)$, i.e. $\left|\tilde{y} - \tilde{\gamma}\left(\tilde{t}^{(1)}\right)\right|_2 = O(\varepsilon)$. Now by Proposition 11 the set of students with ranks in $(y, \gamma(0)]$ under $\mathcal{E}$ and ranks in $(\tilde{y}, \gamma(0)]$ under $\tilde{\mathcal{E}}$ who are assigned to 1 under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ has $\tilde{\eta}$-measure $O(\varepsilon |C|)$. Hence the residual capacity of school 1 at $\tilde{y}$ under $TTC(\tilde{\gamma})$ is $O(\varepsilon |\mathcal{C}|)$, and so since $\tilde{\eta}$ has full support and has density bounded from above and below by M and m, it holds that $\left|\tilde{y} - \tilde{\gamma}\left(\tilde{t}^{(1)}\right)\right|_2 = O\left(\frac{M}{m}\varepsilon |\mathcal{C}|\right)$. (If the residual capacity is negative we can exchange the roles of γ and $\tilde{\gamma}$ and argue similarly.)

Let us now show that the inductive assumption holds. Fix a school c. Then by Proposition 11 the set of students with ranks in $(y, \gamma(0)]$ under $\mathcal{E}$ and ranks in $(\tilde{y}, \gamma(0)]$ under $\tilde{\mathcal{E}}$ who are assigned to c under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ has $\tilde{\eta}$-measure $O(\varepsilon |C|)$. Moreover, since $\left|\tilde{y} - \tilde{\gamma}\left(\tilde{t}^{(1)}\right)\right|_2 = O\left(\frac{M}{m}\varepsilon |\mathcal{C}|\right)$ and $\tilde{\eta}$ has full support and has density bounded from above and below by M and m, the set of students with ranks in $(\tilde{y}, \tilde{\gamma}\left(\tilde{t}^{(1)}\right)]$ assigned to school c by $TTC(\tilde{\gamma})$ has $\tilde{\eta}$-measure $O(\varepsilon |\mathcal{C}|)$. Hence the set of students assigned to c under $TTC(\gamma)$ by time $t^{(1)}$ and not under $TTC(\tilde{\gamma})$ by time $\tilde{t}^{(1)}$ has η-measure $O(\varepsilon |\mathcal{C}|)$. Moreover, if $t^{(1)} < t^{(2)}$ then for sufficiently small ε it holds that $\tilde{t}^{(1)} = \min_c \tilde{t}^{(c)}$, and otherwise there exists a relabeling of the schools such that this is true, and so $\sigma(1) = 1$.

We now show the inductive step, proving for $\ell + 1$ assuming true for $1, 2, \ldots, \ell$. By inductive assumption, for all c the measure of students assigned to c under $TTC(\gamma)$

and not under $TTC(\tilde{\gamma})$ by the points $\gamma\left(t^{(\ell)}\right), \tilde{\gamma}\left(\tilde{t}^{(\ell)}\right)$ is $O(\varepsilon\ell|\mathcal{C}|)$ for all c.

Let $x = \gamma\left(t^{(\ell)}\right)$ and $y = \gamma\left(t^{(\ell+1)}\right)$. Define $\tilde{x} \in Im(\tilde{\gamma})$ to be the minimal point such that $x \le \tilde{x}$ and there exists b such that $x_b = \tilde{x}_b$. We show that $\tilde{x}$ is near $\tilde{\gamma}\left(\tilde{t}^{(\ell)}\right)$, i.e. $\left|\tilde{x} - \tilde{\gamma}\left(\tilde{t}^{(\ell)}\right)\right|_2 = O(\varepsilon)$. Now by inductive assumption $\eta\left(\left\{\theta \mid r^\theta \in \left(x = \gamma\left(t^{(\ell)}\right), \tilde{\gamma}\left(\tilde{t}^{(\ell)}\right)\right]\right\}\right) = O(\varepsilon\ell|\mathcal{C}|)$ and so $\left|x - \tilde{\gamma}\left(\tilde{t}^{(\ell)}\right)\right|_2 = O(\varepsilon)$. Moreover $\left|\tilde{x}_b - \tilde{\gamma}_b\left(\tilde{t}^{(\ell)}\right)\right|_2 = \left|x_b - \tilde{\gamma}_b\left(\tilde{t}^{(\ell)}\right)\right|_2$ which we have just shown is $O(\varepsilon)$. Finally, since η has full support and has density bounded from above and below by M and m, it holds that $\max_{b,c,\tau} \frac{\gamma_b'(\tau)}{\gamma_c'(\tau)} = O\left(\frac{M}{m}\right)$ and so for all c it holds that $\left|\tilde{x}_c - \tilde{\gamma}_c\left(\tilde{t}^{(\ell)}\right)\right| \le O\left(\frac{M}{m}\varepsilon\right)$.

The remainder of the proof runs much the same as in the base case, with slight adjustments to account for the fact that $x \neq \tilde{x}$. Define $\tilde{y} \in Im(\tilde{\gamma})$ to be the minimal point such that $y \le \tilde{y}$ and there exists c such that $y_c = \tilde{y}_c$. We show that $\tilde{y}$ is near $\tilde{\gamma}\left(\tilde{t}^{(\ell+1)}\right)$, i.e. $\left|\tilde{y} - \tilde{\gamma}\left(\tilde{t}^{(\ell+1)}\right)\right|_2 = O(\varepsilon)$. Now by Proposition 11 the set of students with ranks in $(y, x]$ under $\mathcal{E}$ and ranks in $(\tilde{y}, \tilde{x}]$ under $\tilde{\mathcal{E}}$ who are assigned to $\ell+1$ under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ has $\tilde{\eta}$-measure $O(\varepsilon|C|)$. This, together with the inductive assumption that the difference in students assigned to school ℓ is $O(\varepsilon\ell|\mathcal{C}|)$, shows that the residual capacity of school $\ell+1$ at $\tilde{y}$ under $TTC(\tilde{\gamma})$ is $O(\varepsilon(\ell+1)|\mathcal{C}|)$, and so since $\tilde{\eta}$ has full support and has density bounded from above and below by M and m, it holds that $\left|\tilde{y} - \tilde{\gamma}\left(\tilde{t}^{(\ell+1)}\right)\right|_2 = O\left(\frac{M}{m}\varepsilon(\ell+1)|\mathcal{C}|\right)$. (If the residual capacity is negative we can exchange the roles of γ and $\tilde{\gamma}$ and argue similarly.)

Let us now show that the inductive assumption holds. Fix a school c. Then by Proposition 11 the set of students with ranks in $(y, x]$ under $\mathcal{E}$ and ranks in $(\tilde{y}, \tilde{x}]$ under $\tilde{\mathcal{E}}$ who are assigned to c under $TTC(\gamma)$ and not under $TTC(\tilde{\gamma})$ has $\tilde{\eta}$-measure $O(\varepsilon|C|)$. Moreover, since $\left|\tilde{y} - \tilde{\gamma}\left(\tilde{t}^{(\ell+1)}\right)\right|_2 = O\left(\frac{M}{m}\varepsilon(\ell+1)|\mathcal{C}|\right)$ and $\tilde{\eta}$ has full support and has density bounded from above and below by M and m, the set of students with ranks in $\left(\tilde{y}, \tilde{\gamma}\left(\tilde{t}^{(\ell+1)}\right)\right]$ assigned to school c by $TTC(\tilde{\gamma})$ has $\tilde{\eta}$-measure $O(\varepsilon(\ell+1)|\mathcal{C}|)$. Hence the set of students assigned to c under $TTC(\gamma)$ by time $t^{(\ell+1)}$ and not under $TTC(\tilde{\gamma})$ by time $\tilde{t}^{(\ell+1)}$ has η-measure $O(\varepsilon(\ell+1)|\mathcal{C}|)$. Moreover if $t^{(\ell+1)} < t^{(\ell+2)}$ then for sufficiently small ε it holds that $\tilde{t}^{(\ell+1)} = \min_{c>\ell} \tilde{t}^{(c)}$, and otherwise there exists a relabeling of the schools such that this is true, and so $\sigma(\ell+1) = \ell+1$. □

D.6 Proof of Proposition 3

Throughout the proof, we omit the dependence on E. We show that there exist TTC cutoffs $\left\{(\boldsymbol{p}^*)_b^c = \tilde{\gamma}_b\left(t^{(c)}\right)\right\}$ such that the TTC path and stopping times $\gamma, \{\widetilde{t^{(c)}}\}_{c\in\mathcal{C}}$

satisfy trade balance and capacity for $\Phi(E)$ and $B^*(s) \subseteq B(s) \subseteq B(s;\boldsymbol{p^*}) \subseteq B^*(s)$. For brevity, for a TTC path γ and discrete economy E', we say that γ *is a TTC path for* E' if there exist stopping times $\left\{t^{(c)}\right\}_{c\in\mathcal{C}}$ such that $\gamma(\cdot), \left\{t^{(c)}\right\}_{c\in\mathcal{C}}$ satisfy trade balance and capacity for $\Phi(E')$, and write $\boldsymbol{p} = \left\{\gamma_b\left(t^{(c)}\right)\right\} \in \mathcal{P}(E')$.

We first show that $B^*(s) \subseteq B(s)$. Suppose $c \notin B(s)$. Then there exists a TTC path γ for E such that $r^s + \frac{1}{|\mathcal{S}|}\mathbf{1} \leq \gamma\left(t^{(c)}\right)$. Hence for all $\tilde{\succ}$ there exists a TTC path $\tilde{\gamma} \in \mathcal{P}([E_{-s};\tilde{\succ}])$ such that $r^s + \frac{1}{|\mathcal{S}|}\mathbf{1} \leq \tilde{\gamma}\left(t^{(c)}\right)$. By Proposition 2 and Theorem 2 for all $\tilde{\succ}$ it holds that $\mu_{dTTC}(s \mid [E_{-s};\tilde{\succ}]) = \max_{\tilde{\succ}}\left\{c : r^s_b \geq \tilde{\gamma}\left(t^{(c)}\right)_b \text{ for some } b\right\}$. Hence for all $\tilde{\succ}$ it holds that $\mu_{dTTC}(s \mid [E_{-s};\tilde{\succ}]) \neq c$ and so $c \notin B^*(s)$.

We next show that $B(s) \subseteq B(s;\boldsymbol{p^*}) \subseteq B^*(s)$. Intuitively, we construct the special TTC path $\tilde{\gamma}$ for E by clearing as many cycles as possible that do not involve student s. Formally, let $\rhd$ be an ordering over subsets of $\mathcal{C}$ where: (1) all subsets involving student s's top choice available school c (under the preferences $\succ^s$ in E) come after all subsets not involving c; and (2) subject to this, subsets are ordered via the shortlex order. Let $\tilde{\gamma}$ be the TTC path for E obtained by selecting valid directions with minimal support under the order $\rhd$. (Such a path exists since the resulting valid directions d are piecewise Lipschitz continuous.)

It follows trivially from the definition of $B(s)$ that $B(s) \subseteq B(s;\boldsymbol{p^*})$. We now show that $B(s;\boldsymbol{p^*}) \subseteq B^*(s)$. For suppose $c \in B(s;\boldsymbol{p^*})$. Consider the preferences $\succ'$ that put school c first, and then all other schools in the order given by $\succ^s$. We show that $\mu_{dTTC}(s \mid [E_{-s};\succ']) = c$. Now since $c \in B(s;\boldsymbol{p^*})$, it holds that $r^s \not\leq \tilde{\gamma}\left(t^{(c)}\right)$. In other words, if we let $\tau^s = \inf\{\tau \mid \tilde{\gamma}(\tau) \ngeq r^s\}$ be the time that the cube I^s corresponding to student s starts clearing, then school c is available at time τ^s. Moreover if we let $\tilde{\gamma'}$ be the TTC path for $[E_{-s};\succ']$ obtained by selecting valid directions with minimal support under the order $\rhd$, then for all $\tau \leq \tau^s$ it holds that $\tilde{\gamma'}(\tau) = \tilde{\gamma}(\tau)$, and so school c is again available at time τ^s. Hence by Proposition 2 and Theorem 2 it holds that $\mu_{dTTC}(s \mid [E_{-s};\succ']) = c$ and so $c \in B^*(s)$.

E Proofs for Applications (Section 4)

Throughout this section, we will say that a vector d is a *valid direction at point* x if d satisfies the marginal trade balance equations at x, and $d \cdot \mathbf{1} = -1$.

E.1 Optimal Investment in School Quality

In this section, we prove the results stated in Section 4.1. We will assume that the total measure of students is 1, and speak of student measures and student proportions interchangeably.

Proofs for Section 4.1

Proof of Proposition 4. Let $\gamma, p, \left\{t^{(1)}, t^{(2)}\right\}$ be the TTC path, cutoffs and stopping times with quality δ, and let $\hat{\gamma}, \hat{p}, \left\{\hat{t}^{(1)}, \hat{t}^{(2)}\right\}$ be the TTC path, cutoffs and stopping times with quality $\hat{\delta}$. When we change δ_ℓ to $\hat{\delta}_\ell$, this increases the relative popularity of school ℓ.

Consider first when $\ell = 1$. As there are only two schools, $|d_1(x)|$ decreases and $|d_2(x)|$ increases for all x. It follows that if $\gamma_1(t) = \hat{\gamma}_1\left(\hat{t}\right)$ then $\gamma_2(t) \geq \hat{\gamma}_2\left(\hat{t}\right)$, and if $\gamma_2(t) = \hat{\gamma}_2\left(\hat{t}\right)$ then $\gamma_1(t) \leq \hat{\gamma}_1\left(\hat{t}\right)$. Suppose that $p_2^1 \leq \hat{p}_2^1$. Then there exists $t \leq t^{(1)}$ such that $p_2^1 = \gamma_2\left(t^{(1)}\right) \leq \gamma_2(t) = \hat{\gamma}_2\left(\hat{t}^{(1)}\right)$, and so

$$p_1^1 = \gamma_1\left(t^{(1)}\right) \leq \gamma_1(t) \leq \hat{\gamma}_1\left(\hat{t}^{(1)}\right) = \hat{p}_1^1$$

as required. Hence it suffices to show that $p_2^1 \leq \hat{p}_2^1$.

Suppose for the sake of contradiction that $p_2^1 > \hat{p}_2^1$. Then there exists $t < \hat{t}^{(1)}$ such that $p_2^1 = \gamma_2\left(t^{(1)}\right) = \hat{\gamma}_2(t) > \hat{\gamma}_2\left(\hat{t}^{(1)}\right)$, and so $\mathcal{T}_2\left(\gamma; t^{(1)}\right) \subseteq \mathcal{T}_2(\hat{\gamma}; t) \subset \mathcal{T}_2\left(\hat{\gamma}; \hat{t}^{(1)}\right)$ and similarly $\mathcal{T}_1\left(\gamma; t^{(1)}\right) \supset \mathcal{T}_1\left(\hat{\gamma}; \hat{t}^{(1)}\right)$. It follows that

$$\eta\left(\left\{\theta \in \mathcal{T}_2\left(\gamma; t^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 1\right\}\right) < \hat{\eta}\left(\left\{\theta \in \mathcal{T}_2\left(\hat{\gamma}; \hat{t}^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 1\right\}\right),$$

since the set increased and more students want school 1, and similarly

$$\eta\left(\left\{\theta \in \mathcal{T}_1\left(\gamma; t^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 2\right\}\right) > \hat{\eta}\left(\left\{\theta \in \mathcal{T}_1\left(\hat{\gamma}; \hat{t}^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 2\right\}\right),$$

However, integrating over the marginal trade balance equations gives that

$$\eta\left(\left\{\theta \in \mathcal{T}_2\left(\gamma; t^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 1\right\}\right) = \eta\left(\left\{\theta \in \mathcal{T}_1\left(\gamma; t^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 2\right\}\right) \text{ and}$$
$$\hat{\eta}\left(\left\{\theta \in \mathcal{T}_2\left(\hat{\gamma}; \hat{t}^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 1\right\}\right) = \hat{\eta}\left(\left\{\theta \in \mathcal{T}_1\left(\hat{\gamma}; \hat{t}^{(1)}\right) \mid \max_{\succ^\theta}\{1,2\} = 2\right\}\right),$$

which gives the required contradiction. The fact that p_2^2 decreases follows from the fact that p_1^1 increases, since the total number of assigned students is the same. $\square$

Proof of Proposition 5.

TTC Cutoffs We calculate the TTC cutoffs under the logit model for different student choice probabilities by using the TTC paths and trade balance equations. In round 1, the marginals $\tilde{H}_b^c(x)$ for $b, c \in \mathcal{C}$ at each point $x \in [0,1]$ are given by $\tilde{H}_b^c(x) = e^{\delta_c} \prod_{c' \neq b} x_{c'}$. Hence $v_b = \sum_c \tilde{H}_b^c(x) = \left(\sum_b e^{\delta_c}\right) \prod_{c' \neq b} x_{c'} = \prod_{c' \neq b} x_{c'}$, so $v = \frac{\prod_c x_c}{\min_c x_c}$ and the matrix $H(x)$ is given by

$$H_{b,c}(x) = e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} + \mathbf{1}_{b-c}\left(1 - \frac{\min_{c'} x_{c'}}{x_b}\right) = \begin{cases} 1 - \left(1 - e^{\delta_c}\right) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise,} \end{cases}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x) H(x) = d(x)$. To solve for this, we observe that this equation is the same as $d(x)(H(x) - I) = \mathbf{0}$, where I is the n-dimensional identity matrix, and and $[H(x) - I]$ has (b,c)th entry

$$[H(x) - I]_{b,c} = \begin{cases} -\left(1 - e^{\delta_b}\right) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise.} \end{cases}$$

Since this has rank $n-1$, the nullspace is easily obtained by replacing the last column of $H(x) - I$ with ones, inverting the matrix and left multiplying it to the vector $e^{|\mathcal{C}|}$ (the vector with all zero entries, other than a 1 in the $|\mathcal{C}|$th entry). This yields the valid direction $d(x)$ with cth component

$$d_c(x) = -\frac{e^{\delta_c} x_c}{\sum_b e^{\delta_b} x_b}.$$

We now find a valid TTC path γ using the trade balance equations 1. Since the ratios of the components of the gradient $\frac{d_b(x)}{d_c(x)}$ only depend on x_b, x_c and the $\delta_{c'}$, we solve for x_c in terms of x_1, using the fact that the path starts at $(1,1)$. This gives the path γ defined by $\gamma_c\left(\gamma_1^{-1}(x_1)\right) = x_1^{e^{\delta_c - \delta_1}}$ for all c.

Recall that the schools are indexed so that school c_1 is the most demanded school, that is, $\frac{e^{\delta_1}}{q_1} = \max_c \frac{e^{\delta_c}}{q_c}$. Since we are only interested in the changes in the cutoffs $\gamma\left(t^{(1)}\right)$ and not in the specific time, let us assume without loss of generality that

$\gamma_1(t) = 1 - t$. Then school c_1 fills at time $t^{(1)} = 1 - \left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_1}}{\sum_{c'} e^{\delta_{c'}}}} = 1 - \left(1 - \rho_1\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_1}}{\sum_{c'} e^{\delta_{c'}}}}$. Hence the round 1 cutoffs are

$$p_b^1 = \left(1 - t^{(1)}\right)^{e^{\delta_b - \delta_1}} = \left(1 - \rho_1\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}} = \left(1 - \rho_1\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\pi^{b|c}}. \tag{10}$$

It can be shown by projecting onto the remaining coordinates and using induction that the round i cutoffs are given by

$$p_b^c = \begin{cases} \left(\prod_{c'<c} \frac{1}{p_{c'}^{c'}}\right)^{\pi^{b|c}} \left(\prod_{c'} p_{c'}^{c-1} - \rho_c\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\pi^{b|c}} & \text{if } b \geq c \\ p_b^b & \text{if } b \leq c. \end{cases}$$

TTC Cutoffs - Comparative Statics We perform some comparative statics calculations for the TTC cutoffs under the logit model. For $b \neq \ell$ it holds that the TTC cutoff p_b^1 for using priority at school b to receive a seat at school 1 is decreasing in δ_ℓ,

$$\begin{aligned} \frac{\partial p_b^1}{\partial \delta_\ell} &= \frac{\partial}{\partial \delta_\ell}\left[\left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'} e^{\delta_{c'}}}}\right] \\ &= -p_b^1\left(\frac{e^{\delta_\ell+\delta_b}}{\left(\Delta^1\right)^2}\right)\left[-\ln\left(\frac{1}{1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1}\right) + \frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)} - 1\right] \end{aligned}$$

is negative, since $0 < \frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)} < 1$, where for brevity we define $\Delta^c = \sum_{b\geq c} e^{\delta_b}$.

We can decompose this change as

$$\frac{\partial p_b^1}{\partial \delta_\ell} = -p_b^1\left(\frac{e^{\delta_\ell+\delta_b}}{\left(\Delta^1\right)^2}\right)\left[\ln\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)\right] - p_b^1\left(\frac{e^{\delta_\ell+\delta_b}}{\left(\Delta^1\right)^2}\right)\left[\frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)} - 1\right] < 0,$$

where the first term is the increase in p_b^1 due to the fact that relatively fewer students are pointed to and cleared by school b for every marginal change in rank, and the second term is the decrease in p_b^1 due to the fact that school 1 is relatively less popular now, and so more students need to be given a budget set of $\mathcal{C}^{(1)}$ in order for school 1 to reach capacity.

For $b = \ell$ the TTC cutoff p^1_ℓ is again decreasing in δ_ℓ,

$$\frac{\partial p^1_\ell}{\partial \delta_\ell} = \frac{\partial}{\partial \delta_\ell}\left[\left(1 - \frac{q_1}{e^{\delta_1}}\left(\sum_{c'} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_\ell}}{\sum_{c'} e^{\delta_{c'}}}}\right]$$
$$= -p^1_b\left(\frac{e^{\delta_\ell}\left(\Delta^1 - e^{\delta_\ell}\right)}{\left(\Delta^1\right)^2}\right)\ln\left(\frac{1}{1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1}\right) - p^1_b\left(\frac{e^{2\delta_\ell+\delta_b}}{\left(\Delta^1\right)^2}\right)\left(\frac{1}{\left(1 - \left(\frac{q_1}{e^{\delta_1}}\right)\Delta^1\right)} - 1\right)$$

is negative since both terms are negative.

Similarly, for $c < \ell$ and $b \neq \ell$ the TTC cutoff p^c_ℓ is decreasing in δ_ℓ, since (if we let $\tilde{q}_c = \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}$)

$$\frac{\partial p^c_b}{\partial \delta_\ell} = \frac{\partial}{\partial \delta_\ell}\left[\left(1 - \left(\prod_{c'<c}\frac{1}{p^{c'}_{c'}}\right)\tilde{q}_c\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'\geq c} e^{\delta_{c'}}}}\right]$$
$$= -p^c_b\left(\frac{e^{\delta_\ell+\delta_b}}{\left(\Delta^c\right)^2}\right)\left[-\ln\left(\frac{1}{1 - P^c\tilde{q}_c\Delta^c}\right) + \frac{1}{\left(1 - P^c\tilde{q}_c\Delta^c\right)} - 1\right] - p^c_b\left[\frac{e^{\delta_b}\tilde{q}_c\frac{\partial P^c}{\partial \delta_\ell}}{\left(1 - P^c\tilde{q}_c\Delta^c\right)}\right]$$

is negative, where $P^c = \prod_{c'<c}\frac{1}{p^{c'}_{c'}}$, since $0 < 1 - P^c\tilde{q}_c\Delta^c < 1$ and $\frac{\partial P^c}{\partial \delta_\ell} = P^c\left(\sum_{c'<c}\frac{-\frac{\partial p^{c'}_{c'}}{\partial \delta_\ell}}{p^{c'}_{c'}}\right) > 0$ so both terms are negative.

We can decompose this change as follows. Let $P^c = \prod_{c'<c}\frac{1}{p^{c'}_{c'}}$. For $c < \ell$ and $b \geq c$, $b \neq \ell$ it holds that

$$\frac{\partial p^c_b}{\partial \delta_\ell} = -p^c_b\left(\frac{e^{\delta_\ell+\delta_b}}{\left(\Delta^c\right)^2}\right)\left[\ln\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)\right]$$
$$- p^c_b\left(\frac{e^{\delta_\ell+\delta_b}}{\left(\Delta^c\right)^2}\right)\left[\frac{1}{\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)} - 1\right] - p^c_b\left[\frac{e^{\delta_b}\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\frac{\partial}{\partial \delta_\ell}P^c}{\left(1 - P^c\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\Delta^c\right)}\right]$$

which is negative. The first term is the increase in p^c_b due to the fact that relatively fewer students are pointed to and cleared by school j for every marginal change in rank, and the second and third terms are the decrease in p^c_b due to the fact that schools 1 through c are relatively less popular now, and so more students need to be given a budget set of $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(c)}$ in order for schools 1 through c to reach capacity.

For $c < \ell$ and $b = \ell$ the TTC cutoff p^c_ℓ is also decreasing in δ_ℓ, since (if we let

$\tilde{q}_c = \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}})$

$$\begin{aligned}\frac{\partial p_\ell^c}{\partial \delta_\ell} =& \frac{\partial}{\partial \delta_\ell}\left[\left(1-\left(\prod_{c'<c}\frac{1}{p_{c'}^{c'}}\right)\tilde{q}_c\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_\ell}}{\sum_{c'\geq c} e^{\delta_{c'}}}}\right] \\ =& -p_\ell^c\left[\left(\frac{e^{\delta_\ell}\left(\Delta^c - e^{\delta_\ell}\right)}{\left(\Delta^c\right)^2}\right)\ln\left(\frac{1}{1-P^c\tilde{q}_c\Delta^c}\right)\right] - p_\ell^c\left(\frac{e^{\delta_\ell}}{\Delta^c}\right)\left[\frac{\left(P^c\tilde{q}_c e^{\delta_\ell}\right)+\tilde{q}_c\Delta^c\frac{\partial P^c}{\partial \delta_\ell}}{\left(1-P^c\tilde{q}_c\Delta^c\right)}\right]\end{aligned}$$

which is negative, since $\frac{\partial P^c}{\partial \delta_\ell} = P^c\left(\sum_{c'<c} -\frac{\partial p_{c'}^{c'}}{\partial \delta_\ell}\cdot\frac{1}{p_{c'}^{c'}}\right) > 0$ so both terms are negative.

When $c = \ell$, the effects of changing δ_ℓ on the cutoffs required the obtain a seat at school ℓ are a little more involved. For $c = \ell$ and $b \neq \ell$,

$$\begin{aligned}\frac{\partial p_b^\ell}{\partial \delta_\ell} =& \frac{\partial}{\partial \delta_\ell}\left[\left(1-\left(\prod_{c'<\ell}\frac{1}{p_{c'}^{c'}}\right)\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_b}}{\sum_{c'\geq \ell} e^{\delta_{c'}}}}\right] \\ =& p_b^\ell\left(\frac{e^{\delta_b}}{\Delta^\ell}\right)\left[\left(\frac{e^{\delta_\ell}}{\Delta^\ell}\right)\ln\left(\frac{1}{1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell}\right)+\frac{\frac{\partial}{\partial \delta_\ell}\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}{\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}\right]\end{aligned}$$

where $P_\ell = \prod_{c'<\ell}\frac{1}{p_{c'}^{c'}}$, the first term is positive, and the second term has the same sign as its numerator $\frac{\partial}{\partial \delta_\ell}\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)$. Similarly for $c = \ell$ and $b = \ell$,

$$\begin{aligned}\frac{\partial p_\ell^\ell}{\partial \delta_\ell} =& \frac{\partial}{\partial \delta_\ell}\left[\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\left(\sum_{c'\geq c} e^{\delta_{c'}}\right)\right)^{\frac{e^{\delta_\ell}}{\sum_{c'\geq \ell} e^{\delta_{c'}}}}\right] \\ =& p_\ell^\ell\left(\frac{e^{\delta_b}}{\Delta^\ell}\right)\left[-\left(\frac{\Delta^\ell - e^{\delta_\ell}}{\Delta^\ell}\right)\ln\left(\frac{1}{1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell}\right)+\frac{\frac{\partial}{\partial \delta_\ell}\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}{\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)}\right]\end{aligned}$$

where $P_\ell = \prod_{c'<\ell}\frac{1}{p_{c'}^{c'}}$, the first term is negative, and the second term has the same sign as its numerator $\frac{\partial}{\partial \delta_\ell}\left(1-P_\ell\left(\frac{q_\ell}{e^{\delta_\ell}}-\frac{q_{\ell-1}}{e^{\delta_{\ell-1}}}\right)\Delta^\ell\right)$. Since $\frac{\partial}{\partial \delta_\ell}\left(\prod_{b\geq\ell} p_b^\ell\right) > 0$, it follows that $\frac{\partial p_b^\ell}{\partial \delta_\ell} > 0$ for all $b \neq \ell$, and there are regimes in which $\frac{\partial p_\ell^\ell}{\partial \delta_\ell}$ is positive, and regimes where it is negative. □

Proofs for Section 4.1

Proof of Proposition 6.

Welfare Expressions We derive the welfare expressions corresponding to these cutoffs. Let $\mathcal{C}^{(c)} = \{c, c+1, \ldots, n\}$. Since the schools are ordered so that $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \cdots \leq \frac{q_n}{e^{\delta_n}}$, it follows that the schools also fill in the order $1, 2, \ldots, n$.

Suppose that the total mass of students is 1. Then the mass of students with budget set $\mathcal{C}^{(1)}$ is given by $N^1 = q_1 \left(\frac{\sum_b e^{\delta_b}}{e^{\delta_1}}\right)$, and the mass of students with budget set $\mathcal{C}^{(2)}$ is given by $N^2 = \left(q_2 - \frac{e^{\delta_2}}{\sum_b e^{\delta_b}} N^1\right)\left(\frac{\sum_{b\geq 2} e^{\delta_b}}{e^{\delta_2}}\right) = \left(\frac{q_2}{e^{\delta_2}} - \frac{q_1}{e^{\delta_1}}\right)\left(\sum_{b\geq 2} e^{\delta_b}\right)$. An inductive argument shows that the proportion of students with budget set $\mathcal{C}^{(c)}$ is

$$N^c = \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right)\left(\sum_{b\geq c} e^{\delta_b}\right).$$

which depends only on δ_b for $b \geq c-1$.

Moreover, each such student with budget set $\mathcal{C}^{(c)}$, conditional on their budget set, has expected utility Small and Rosen (1981)

$$U^c = \mathbb{E}\left[\max_{c' \in \mathcal{C}^{(c)}} \{\delta_b + \varepsilon_{\theta c'}\}\right] = \ln\left[\sum_{b\geq c} e^{\delta_b}\right],$$

which depends only on δ_b for $b \geq c$. Hence the expected social welfare from fixed qualities δ_c is given by

$$U_{TTC} = \sum_c N^c \cdot U^c = \sum_c \left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right) \Delta^c \ln \Delta^c,$$

where $\Delta^c = \sum_{b\geq c} e^{\delta_b}$.

Welfare - Comparative Statics Taking derivatives, we obtain that

$$\frac{dU_{TTC}}{d\delta_\ell} = \sum_c \left(\frac{dN^c}{d\delta_\ell} \cdot U^c + N^c \cdot \frac{dU^c}{d\delta_\ell}\right) = \sum_{c\leq \ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c + \sum_{c\leq \ell} N^c \cdot \frac{dU^c}{d\delta_\ell},$$

where $\sum_{c\leq\ell} N^c \cdot \frac{dU^c}{d\delta_\ell} = e^{\delta_\ell} \sum_{c\leq\ell}\left(\frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}\right) = q_\ell$. It follows that

$$\frac{dU_{TTC}}{d\delta_\ell} = q_\ell + \sum_{c\leq\ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c.$$

□

Proof of Proposition 7. We solve for the social welfare maximixing budget allocation. For a fixed runout ordering (i.e. $\frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \cdots \leq \frac{q_n}{e^{\delta_n}}$), the central school board's investment problem is given by the program

$$\begin{aligned} \max_{\kappa_1,\kappa_2,\ldots,\kappa_n} & \sum_i \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_i}\right)\left(\sum_{j\geq i}\kappa_j\right)\ln\left(\sum_{j\geq i}\kappa_j\right) \\ s.t. & \frac{q_{i-1}}{\kappa_{i-1}} \leq \frac{q_i}{\kappa_{i-1}} \ \forall i \\ & \sum_i \kappa_i = K \\ & q_0 = 0. \end{aligned} \tag{11}$$

We can reformulate this as the following program,

$$\max_{\kappa_2,\ldots,\kappa_n} \left(\frac{q_1}{K-\sum_i \kappa_i}\right) K\ln K + \left(\frac{q_2}{\kappa_2} - \frac{q_1}{K-\sum_i\kappa_i}\right) U_2 \ln U_2 + \sum_{i\geq 3}\left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}}\right) U_i \ln U_i \tag{12}$$

$$\begin{aligned} s.t. \quad & \frac{q_{i-1}}{\kappa_{i-1}} \leq \frac{q_i}{\kappa_i} \ \forall i \geq 3 \\ & \frac{q_1}{K-\sum_i\kappa_i} \leq \frac{q_2}{\kappa_2}, \end{aligned}$$

where $U_i = \sum_{j\geq i}\kappa_j$.

The reformulated problem (12) has objective function

$$U(\boldsymbol{\kappa}) = \left(\frac{q_1}{K-\sum_i\kappa_i}\right) K\ln K + \left(\frac{q_2}{\kappa_2} - \frac{q_1}{K-\sum_i\kappa_i}\right)\Delta^2\ln\Delta^2 + \sum_{i\geq 3}\left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}}\right)\Delta^i\ln\Delta^i,$$

where $\Delta^i = \sum_{j\geq i}\kappa_j$. Taking the derivatives with respect to the budget allocations κ_k gives

$$\frac{\partial U}{\partial\kappa_k} = \left(\frac{q_1}{\left(K-\sum_i\kappa_i\right)^2}\right)\left(K\ln\frac{K}{\Delta^2} - \left(K-\Delta^2\right)\right) + \sum_{2\leq i<k}\frac{q_i}{\kappa_i}\ln\frac{\Delta^i}{\Delta^{i+1}} + \frac{q_k}{(\kappa_k)^2}\left(\kappa_k - \Delta^{k+1}\ln\frac{\Delta^k}{\Delta^{k+1}}\right),$$

where $K\ln\frac{K}{\Delta^2} - (K-\Delta^2) \geq 0$, $\ln\frac{\Delta^i}{\Delta^{i+1}} \geq 0$,fff and $\kappa_k - \Delta^{k+1}\ln\frac{\Delta^k}{\Delta^{k+1}} \geq 0$ and so

$\frac{\partial U}{\partial \kappa_k} \geq 0 \, \forall k$.

Moreover, if $\frac{q_{i-1}}{\kappa_{i-1}} = \frac{q_i}{\kappa_i}$, then defining a new problem with $n-1$ schools, and capacities $\tilde{q}$ and budget $\tilde{\kappa}$

$$\tilde{q_j} = \begin{cases} q_j & \text{if } j < i-1 \\ q_{i-1} + q_i & \text{if } j = i-1 \\ q_{j+1} & \text{if } j > i-1 \end{cases}, \; \tilde{\kappa_j} = \begin{cases} \kappa_j & \text{if } j < i-1 \\ \kappa_{i-1} + \kappa_i & \text{if } j = i-1 \\ \kappa_{j+1} & \text{if } j > i-1 \end{cases}$$

leads to a problem with the same objective function, since

$$\begin{aligned} &\left(\frac{q_{i-1}}{\kappa_{i-1}} - \frac{q_{i-2}}{\kappa_{i-2}}\right) \Delta^{i-1} \ln \Delta^{i-1} + \left(\frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}}\right) \Delta^{i} \ln \Delta^{i} + \left(\frac{q_{i+1}}{\kappa_{i+1}} - \frac{q_i}{\kappa_i}\right) \Delta^{i+1} \ln \Delta^{i+1} \\ = &\left(\frac{q_{i-1}+q_i}{\kappa_{i-1}+\kappa_i} - \frac{q_{i-2}}{\kappa_{i-2}}\right) \Delta^{i-1} \ln \Delta^{i-1} + \left(\frac{q_{i+1}}{\kappa_{i+1}} - \frac{q_{i-1}+q_i}{\kappa_{i-1}+\kappa_i}\right) \Delta^{i+1} \ln \Delta^{i+1}. \end{aligned}$$

Hence if there exists i for which $\frac{q_i}{\kappa_i} \neq \frac{q_{i-1}}{\kappa_{i-1}}$, we may take i to be minimal such that this occurs, decrease each of $\kappa_1, \ldots, \kappa_{i-1}$ proportionally so that $\kappa_1 + \cdots + \kappa_{i-1}$ decreases by ε and increase κ_i by ε and increase resulting value of the objective. It follows that the objective is maximized when $\frac{q_1}{\kappa_1} = \frac{q_2}{\kappa_2} = \cdots = \frac{q_n}{\kappa_n}$, i.e. when the money assigned to each school is proportional to the number of seats at the school. □

E.2 Design of TTC Priorities

We demonstrate how to calculate the TTC cutoffs for the two economies in Figure 7 by using the TTC paths and trade balance equations.

Consider the economy $\mathcal{E}_0$, where the top priority students have ranks uniformly distributed in $[m, 1]^2$. If $x = (x_1, x_1)$ is on the diagonal, then $\tilde{H}_i^j(x) = \frac{x_1}{2}$ for all $i, j \in \{1, 2\}$, and so there is a unique valid direction $d(\vec{x}) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Moreover, $\gamma(t) = \left(\frac{t}{2}, \frac{t}{2}\right)$ satisfies $\frac{d\gamma(t)}{dt} = \boldsymbol{d}(\gamma(t))$ for all t and hence Theorem 2 implies that $\gamma(t) = \left(\frac{t}{2}, \frac{t}{2}\right)$ is the unique TTC path, and the cutoff points $p_b^c = \sqrt{1-2q}$ give the unique TTC allocation.

Consider now the economy $\mathcal{E}_1$, where top priority students have ranks uniformly distributed in the $\tilde{r} \times \tilde{r}$ square $(1-\tilde{r}, 1] \times (m, m+\tilde{r}]$ for some small $\tilde{r}$.

If x is in $(1-\tilde{r}, 1] \times [m+\tilde{r}, 1]$ then $H_1^j(x) = \frac{1}{2}\left(m + (1-m)\frac{1-m}{\tilde{r}}\right)$ for all j and

$H_2^j(x) = \frac{m}{2}$ for all j, hence there is a unique valid direction $d(x) = \frac{1}{2+\frac{r^2}{\tilde{r}(1-r)}} \begin{bmatrix} -1 \\ -1 - \frac{r^2}{\tilde{r}(1-r)} \end{bmatrix}$.

If x is in $(m, 1-\tilde{r}] \times (m, 1]$ then $H_i^j(x) = \frac{m}{2}$ for all i, j and there is a unique valid direction $d(x) = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

Finally, if $x = (x_1, x_2)$ is in $[0,1] \setminus (m,1]^2$ then $H_1^j(x) = \frac{1}{2}x_2$ and $H_2^j = \frac{1}{2}x_1$ for all j and there is a unique valid direction $d(x) = \frac{1}{x_1+x_2} \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$. Hence the TTC path $\gamma(t)$ has gradient proportional to $\begin{bmatrix} -1 \\ -1 - \frac{(1-m)^2}{\tilde{r}m} \end{bmatrix}$ from the point $(1,1)$ to the point $\left(1-\tilde{r}, 1-\tilde{r}-\frac{r^2}{1-r}\right)$, to $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ from the point $\left(1-\tilde{r}, 1-\tilde{r}-\frac{r^2}{1-r}\right)$ to the point $\left(m + \frac{r^2}{1-r}, m\right)$ and to $\begin{bmatrix} -1 - \frac{(1-m)^2}{m^2} \\ -1 \end{bmatrix}$ from the point $\left(m + \frac{(1-m)^2}{m}, m\right)$ to the point $\left(\sqrt{\frac{1-2q}{1-2m+2m^2}}, \sqrt{(1-2q)(1-2m+2m^2)}\right) = (\overline{p}, \underline{p})$.

Finally, we show that if economy $\mathcal{E}_2$ is given by perturbing the relative ranks of students in $\left\{\theta \mid r_c^\theta \geq m \ \forall c\right\}$, then the TTC cutoffs for $\mathcal{E}_2$ are given by $p_1^1 = p_1^2 = x,\ p_2^1 = p_2^2 = y$ where $x \leq \overline{p} = \sqrt{\frac{1-2q}{1-2m+2m^2}}$ and $y \geq \underline{p} = \sqrt{(1-2q)(1-2m+2m^2)}$. (By symmetry, it follows that $\underline{p} \leq x, y \leq \overline{p}$.) Let γ_1 and γ_2 be the TTC paths for $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Then the TTC path γ_2 for $\mathcal{E}_2$ has gradient $\frac{1}{x_{bound}+m} \begin{bmatrix} -x_{bound} \\ -m \end{bmatrix}$ from (x_{bound}, m) to (x, y).

Consider the aggregate trade balance equations for students assigned before the TTC path reaches (x_{bound}, m). They stipulate that the measure of students in $[0,m] \times [m,1]$ who prefer school 1 is at most the measure of students who are either perturbed, or in $[x_{bound}, 1] \times [0, m]$, and who prefer school 2. This means that $\frac{1}{2}m(1-m) \leq \frac{1}{2}\left((1-m)^2 + m(1-x_{bound})\right)$, or $x_{bound} \leq m + \frac{(1-m)^2}{m}$, and hence γ_2 lies above γ_1[39] and so $x \leq \overline{p}$ and $y \geq \frac{1-2q}{\overline{p}} = \underline{p}$.

[39] That is, for each x_1, if (x_1, y_1) lies on γ_1 and (x_1, y_2) lies on γ_2, then $y_2 \geq y_1$.

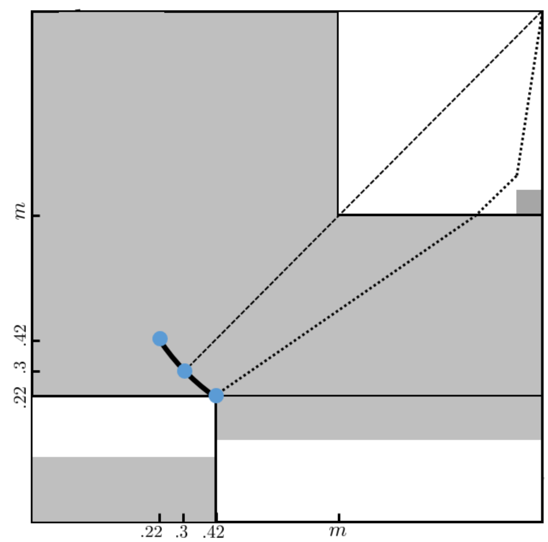

Figure 13: Economy $\mathcal{E}_1$ from Example 9. The black borders partition the space of students into four regions. The density of students is zero on white areas, and constant on each of the shaded areas within a bordered region. In each of the four regions, the total measure of students within is equal to the total area (white and shaded) within the borders of the region.

E.3 Comparing Top Trading Cycles and Deferred Acceptance

In this section, we derive the expressions for the TTC and DA cutoffs given in Section 4.3.

Consider the TTC cutoffs for the neighborhood priority setting. We prove by induction on ℓ that $p_j^\ell = 1 - \frac{q_\ell}{2q}$ for all ℓ, j such that $j \geq \ell$.

Base case: $\ell = 1$.

For each school i, there are measure q of students whose first choice school is i, αq of whom have priority at i and $\frac{(1-\alpha)q}{n-1}$ of whom have priority at school j, for all $j \neq i$.

The TTC path is given by the diagonal, $\gamma(t) = \left(1 - \frac{t}{\sqrt{n}}, 1 - \frac{t}{\sqrt{n}}, \ldots, 1 - \frac{t}{\sqrt{n}}\right)$. At the point $\gamma(t) = (x, x, \ldots, x)$ (where $x \geq \frac{1}{2}$) a fraction $2(1-x)$ of students from each neighborhood have been assigned. Since the same proportion of students have each school as their top choice, this means that the quantity of students assigned to each school i is $2(1-x)q$. Hence the cutoffs are given by considering school 1, which has the smallest capacity, and setting the quantity assigned to school 1 equal to its capacity q_1. It follows that $p_j^1 = x^*$ for all j, where $2(1-x^*)q = q_1$, which yields

$$p_j^1 = 1 - \frac{q_1}{2q} \text{ for all } j.$$

Inductive step.

Suppose we know that the cutoffs $\left\{p_j^i\right\}_{i,j\,:\,i\leq\ell}$ satisfy $p_j^i = 1 - \frac{q_i}{2q}$. We show by induction that the $(\ell+1)$th set of cutoffs $\left\{p_j^{\ell+1}\right\}_{j>\ell}$ are given by $p_j^{\ell+1} = 1 - \frac{q_{\ell+1}}{2q}$.

The TTC path is given by the diagonal when restricted to the last $n-\ell$ coordinates, $\gamma\left(t^{(\ell)}+t\right) = \left(p_1^1, p_2^2, \ldots, p_\ell^\ell, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}, \ldots, p_\ell^\ell - \frac{t}{\sqrt{n-\ell}}\right)$.

Consider a neighborhood i. If $i>\ell$, at the point $\gamma(t) = (p_1^1, p_2^2, \ldots, p_\ell^\ell, x, x, \ldots, x)$ (where $x \geq \frac{1}{2}$) a fraction $2\left(p_\ell^\ell - x\right)$ of (all previously assigned and unassigned) students from neighborhood i have been assigned in round $\ell+1$. If $i \leq \ell$, no students from neighborhood i have been assigned in round $\ell+1$.

Consider the set of students S who live in one of the neighborhoods $\ell+1, \ell+2, \ldots, n$. The same proportion of these students have each remaining school as their top choice out of the remaining schools. This means that for any $i>\ell$, the quantity of students assigned to school i in round $\ell+1$ by time t is a $\frac{1}{n-\ell}$ fraction of the total number of students assigned in round $\ell+1$ by time t, and is given by$(n-\ell)\,q\frac{1}{n-\ell} = 2\left(p_\ell^\ell - x\right)q$. Hence the cutoffs are given by considering school $\ell+1$, which has the smallest residual, and setting the quantity assigned to school $\ell+1$ equal to its residual capacity $q_{\ell+1}-q_\ell$. It follows that $p_j^{\ell+1} = x^*$ for all $j>\ell$ where $2\left(p_\ell^\ell - x^*\right)q = q_{\ell+1}-q_\ell$, which yields

$$p_j^{\ell+1} = p_\ell^\ell - \frac{q_{\ell+1}-q_\ell}{2q} = 1 - \frac{q_\ell}{2q} - \frac{q_{\ell+1}-q_\ell}{2q} = 1 - \frac{q_{\ell+1}}{2q} \text{ for all } j>\ell.$$

This completes the proof that the TTC cutoffs are given by $p_j^i = p_i^j = 1 - \frac{q_i}{2q}$ for all $i \leq j$.

Now consider the DA cutoffs. We show that the cutoffs $p_i = 1 - \frac{q_i}{2q}$ satisfy the supply-demand equations. We first remark that the cutoff at school i is higher than all the ranks of students without priority at school i, $p_i \geq \frac{1}{2}$. Since every student has priority at exactly one school, this means that every student is either above the cutoff for exactly one school and is assigned to that school, or is below all the cutoffs and remains unassigned. Hence there are $2q(1-p_i) = q_i$ students assigned to school i for all i, and the supply-demand equations are satisfied.

F Proofs for Section A

F.1 Derivation of Marginal Trade Balance Equations

In this section, we show that the marginal trade balance equations (1) hold,

$$\gamma'(\tau) = \gamma'(\tau)\,\widetilde{H}(\gamma(\tau)) \text{ for all } \tau.$$

The idea is that the measure of students who trade into a school c must be equal to the measure of students who trade out of c.

In particular, suppose that at some time τ the TTC algorithm has assigned exactly the set of students with rank better than $x = \gamma(\tau)$, and the set of available schools is C. Consider the incremental step of a TTC path γ from $\gamma(\tau) = x$ over ϵ units of time. The process of cycle clearing imposes that for any school $c \in C$, the total amount of seats offered by school c from time τ to $\tau + \epsilon$ is equal to the amount of students assigned to c plus the amount of seats that were offered but not claimed over that same time period. In the continuum model the set of seats offered but not claimed is of η-measure 0.[40] Hence the set of students assigned to school c from time τ to $\tau + \epsilon$ has the same measure as the set of students who were offered a seat at school c in that time,

$$\begin{aligned} &\eta\left(\left\{\theta \in \Theta \mid r^\theta \in [\gamma(\tau+\epsilon), \gamma(\tau)) \text{ and } Ch^\theta(C) = c\right\}\right) \\ =&\eta\left(\left\{\theta \in \Theta \mid \exists \tau' \in [\tau, \tau+\epsilon] \text{ s.t. } r^\theta_c = \gamma_c(\tau') \text{ and } r^\theta \le \gamma(\tau')\right\}\right), \end{aligned} \tag{13}$$

or more compactly, $\eta(\mathcal{T}^c(\gamma; [\tau, \tau+\epsilon])) = \eta(\mathcal{T}_c(\gamma; [\tau, \tau+\epsilon]))$.

Let us divide equation (13) by $\delta_c(\epsilon) = \gamma_c(\tau) - \gamma_c(\tau+\epsilon)$ and take the limit as $\epsilon \to 0$. We will show that the resulting left hand side expression is equal to

$$\begin{aligned} &\sum_{b \in C} \lim_{\epsilon \to 0} \frac{\delta_b(\epsilon)}{\delta_c(\epsilon)} \cdot \frac{1}{\delta_b(\epsilon)} \eta\left(\left\{\theta \in \Theta \mid r^\theta \in [x - \delta_b \cdot e^b, x) \text{ and } Ch^\theta(C) = c\right\}\right) \\ =&\sum_{b \in C} \frac{\gamma'_b(\tau)}{\gamma'_c(\tau)} \cdot H_b^{c|C}(x), \end{aligned} \tag{14}$$

[40] A student can have a seat that is offered but not claimed in one of two ways. The first is the seat is offered at time τ and not yet claimed by a trade. The second is that the student that got offered two or more seats at the same time $\tau' \le \tau$ (and was assigned through a trade involving only one seat). Both of these sets of students are of η-measure 0 under our assumptions.

where e^b denotes the unit vector in the direction of coordinate b, and $\delta_b(\epsilon) = (\gamma_b(\tau) - \gamma_b(\tau + \epsilon))$. Similarly, we will show that the resulting right hand side expression is equal to[41]

$$\lim_{\delta \to 0} \frac{1}{\delta} \eta\left(\left\{\theta \in \Theta \mid r^\theta \in [x - \delta \cdot e^c, x)\right\}\right) = \sum_{a \in C} H_c^{a|C}(x). \tag{15}$$

After equating equations (14) and (15), a little algebra shows that this is equivalent to the marginal trade balance equations (1),

$$\gamma'(\tau) = \gamma'(\tau) \widetilde{H}(\gamma(\tau)).$$

Let us now formally prove that the marginal trade balance equations follow from equation (13). For $b, c \in \mathcal{C}$, $x \in [0,1]^{\mathcal{C}}$,$\alpha \in \mathbb{R}$ we define the set [42]

$$T_b^c(x, \alpha) \doteq \left\{\theta \in \Theta \mid r^\theta \in [x - \alpha e^b, x) \text{ and } Ch^\theta\left(C\left(r^\theta\right)\right) = c\right\}.$$

We may think of $T_b^c(x, \alpha)$ as the set of the next α students on school b's priority list who are unassigned when $\gamma(\tau) = x$, and want school c. We remark that the sets used in the definition of the $H_b^c(x)$ are precisely the sets $T_b^c(x, \alpha)$.

We can use the sets $T_b^c(x, \alpha)$ to approximate the expressions in equation (13) involving $\mathcal{T}_c(\gamma; \cdot)$ and $\mathcal{T}^c(\gamma; \cdot)$.

Lemma 9. *Let $\gamma(\tau) = x$ and for all $\epsilon > 0$ let $\delta(\epsilon) = \gamma(\tau) - \gamma(\tau + \epsilon)$. For sufficiently small ϵ, during the interval $[\tau, \tau + \epsilon]$, the set of students who were assigned to school c is*

$$\mathcal{T}^c(\gamma; [\tau, \tau + \epsilon]) = \bigcup_b T_b^c(x, \delta_b(\epsilon))$$

and the set of students who were offered a seat at school c is

$$\mathcal{T}_c(\gamma; [\tau, \tau + \epsilon]) = \bigcup_d T_c^d(x, \delta_c(\epsilon)) \cup \Delta$$

for some small set $\Delta \subset \Theta$. Further, it holds that $\lim_{\tau \to 0} \frac{1}{\tau} \cdot \eta(\Delta) = 0$, and for any $c \neq$

[41] The fact that the quantities in equation (13) are equal to the quantities in equations (14) and (15) follows from our assumption that the density is bounded, since in both cases we double count a set of students whose ranks have Lebesgue measure tending to 0.

[42] We use the notation $[\underline{x}, \overline{x}) = \{z \in \mathbb{R}^n \mid \underline{x}_i \leq z_i < \overline{x}_i \ \forall i\}$ for $\underline{x}, \overline{x} \in \mathbb{R}^n$, and $e^c \in \mathbb{R}^{\mathcal{C}}$ is a vector whose c-th coordinate is equal to 1 and all other coordinates are 0.

$c', d \neq d' \in \mathcal{C}$ *we have* $\lim_{\tau \to 0} \frac{1}{\tau} \cdot \eta \left(T_c^d \left(x, \delta_c \left(\epsilon \right) \right) \cap T_{c'}^d \left(x, \delta_{c'} \left(\epsilon \right) \right) \right) = 0$ *and* $T_c^d \left(x, \delta_c \left(\tau \right) \right) \cap T_c^{d'} \left(x, \delta_c \left(\epsilon \right) \right) = \phi$.

Proof. The first two equations are easily verified, and the fact that the last intersection is empty is also easy to verify. To show the bound on the measure of Δ, we observe that it is contained in the set $\bigcup_{c'} \cup_d \left(T_c^d(x, \delta_c \left(\epsilon \right)) \cap T_{c'}^d(x, \delta_{c'} \left(\epsilon \right)) \right)$, so it suffices to show that $\lim_{\tau \to 0} \frac{1}{\tau} \cdot \eta \left(T_c^d \left(x, \delta_c \left(\epsilon \right) \right) \cap T_{c'}^d \left(x, \delta_{c'} \left(\epsilon \right) \right) \right) = 0$. This follows from the fact that the density defining η is upper bounded by M, so $\eta \left(T_c^d \left(x, \delta_c \left(\epsilon \right) \right) \cap T_{c'}^d \left(x, \delta_{c'} \left(\epsilon \right) \right) \right) \leq M \left| \gamma_c(\tau) - \gamma_c \left(\tau + \epsilon \right) \right| \left| \gamma_{c'}(\tau) - \gamma_{c'} \left(\tau + \epsilon \right) \right|$. Since for all schools c the function γ_c is continuous and has bounded derivative, it is also Lipschitz continuous, so

$$\frac{1}{\tau} \eta \left(\Delta \right) \leq \frac{1}{\tau} \eta \left(T_c^d \left(x, \delta_c \left(\epsilon \right) \right) \cap T_{c'}^d \left(x, \delta_{c'} \left(\epsilon \right) \right) \right) \leq M L_c L_{c'} \epsilon$$

for some Lipschitz constants L_c and $L_{c'}$ and the lemma follows. □

We now now ready to take limits and verify that equation (13) implies that the marginal trade balance equations hold. Let us divide equation (13) by $\delta_c \left(\epsilon \right) = \gamma_c \left(\tau \right) - \gamma_c \left(\tau + \epsilon \right)$ and take the limit as $\epsilon \to 0$. Then on the left hand side we obtain

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{1}{\delta_c \left(\epsilon \right)} \eta \left(\mathcal{T}^c \left(\gamma; [\tau, \tau + \epsilon] \right) \right) &= \lim_{\epsilon \to 0} \frac{1}{\delta_c \left(\epsilon \right)} \eta \left(\bigcup_b T_b^c \left(x, \delta_b \left(\epsilon \right) \right) \right) \text{ (Lemma 9)} \\
&= \lim_{\epsilon \to 0} \left[\sum_{b \in C} \frac{1}{\delta_c \left(\epsilon \right)} \eta \left(T_b^c \left(x, \delta_b \left(\epsilon \right) \right) \right) + O \left(\frac{\left(\left\| \gamma \left(\tau \right) - \gamma(\tau + \epsilon) \right\|_\infty \right)^2}{\delta_c \left(\epsilon \right)} \right) \right] \; (\nu < M) \\
&= \lim_{\epsilon \to 0} \left[\sum_{b \in C} \frac{1}{\delta_c \left(\epsilon \right)} \eta \left(T_b^c \left(x, \delta_b \left(\epsilon \right) \right) \right) \right] \; (\gamma \text{ Lipschitz continuous}) \\
&= \lim_{\epsilon \to 0} \left[\sum_{b \in C} \frac{\delta_b \left(\epsilon \right)}{\delta_c \left(\epsilon \right)} \cdot \frac{1}{\delta_b \left(\epsilon \right)} \eta \left(\left\{ \theta \in \Theta \mid r^\theta \in [x - \delta_b \left(\epsilon \right) \cdot e^b, x) \text{ and } Ch^\theta \left(C \right) = c \right\} \right) \right] \\
&= \sum_{b \in C} \frac{\gamma_b' \left(\tau \right)}{\gamma_c' \left(\tau \right)} \cdot H_b^{c|C} \left(x \right) \text{ (by definition of } \delta \text{ and } H)
\end{aligned}$$

as required. Similarly, on the right hand side we obtain

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\delta_c(\epsilon)}\eta\left(\mathcal{T}_c(\gamma;[\tau,\tau+\epsilon])\right) &= \lim_{\epsilon\to 0}\left[\sum_{a\in C}\frac{1}{\delta_c(\epsilon)}\eta\left(T_c^a(x,\delta_c(\epsilon))\right)+O\left(\frac{\left(\|\gamma(\tau+\epsilon)-\gamma(\tau)\|_\infty\right)^2}{\delta_c(\epsilon)}\right)\right] \text{ (Lemma 9)}\\
&= \lim_{\epsilon\to 0}\left[\sum_{a\in C}\frac{1}{\delta_c(\epsilon)}\eta\left(T_c^a(x,\delta_c(\epsilon))\right)\right] \text{ (γ is Lipschitz continuous)}\\
&= \lim_{\epsilon\to 0}\left[\sum_{a\in C}\frac{1}{\delta_c(\epsilon)}\eta\left(\left\{\theta\in\Theta \mid r^\theta\in[x-\delta_c(\epsilon)\cdot e^c,x) \text{ and } Ch^\theta(C)=a\right\}\right)\right]\\
&= \sum_{a\in C}H_c^{a|C}(x) \text{ (by definition of δ and H)}
\end{aligned}$$

as required. This completes the proof.

F.2 Proof of Lemma 1

We prove the following slightly more general theorem.

Theorem 5. *Let $\mathcal{E}=(\mathcal{C},\Theta,\eta,q)$ be a continuum economy such that $\widetilde{H}(x)$ is irreducible for all x and C. Then there exists a unique valid TTC path γ. Within each round $\gamma(\cdot)$ is given by*

$$\frac{d\gamma(t)}{dt}=\boldsymbol{d}(\gamma(t))$$

where $\boldsymbol{d}(x)$ is the unique valid direction from $x=\gamma(t)$ that satisfies $\boldsymbol{d}(x)=\boldsymbol{d}(x)\widetilde{H}(x)$.

Moreover, if we let $A(x)$ be obtained from $\widetilde{H}(x)-I$ by replacing the nth column with the all ones vector $\mathbf{1}$, then

$$d(x)=\begin{bmatrix}\mathbf{0}^T & -1\end{bmatrix}A(x)^{-1}.$$

Proof. It suffices to show that $d(\cdot)$ is unique. The existence and uniqueness of $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt}=\boldsymbol{d}(\gamma(t))$ follows by invoking Picard-Lindelöf as in the proof of Theorem 2.

Consider the equations,

$$\begin{aligned}
d(x)\widetilde{H}(x) &= d(x)\\
d(x)\cdot\mathbf{1} &= -1.
\end{aligned}$$

When $\widetilde{H}(x)$ is irreducible, every choice of $n-1$ columns of $\widetilde{H}(x)-I$ gives an independent set whose span does not contain $\mathbf{1}$. Therefore if we let $A(x)$ be given by replacing the nth column in $\widetilde{H}(x) - I$ with $\mathbf{1}$, then $A(x)$ has full rank, and the above equations are equivalent to

$$\begin{aligned} d(x)\, A(x) &= \begin{bmatrix} \mathbf{0}^T & -1 \end{bmatrix}, \\ \text{i.e. } d(x) &= \begin{bmatrix} \mathbf{0}^T & -1 \end{bmatrix} A(x)^{-1}. \end{aligned}$$

Hence $d(x)$ is unique for each x, and hence $\gamma(\cdot)$ is uniquely determined. $\square$